

Areas Between Curves

Section 6.1

Calculus I - Lecture Notes

April 14, 2025

Motivating Question

Can we use integration to measure the area *between* two curves, not just under one?

We've used definite integrals to find the area under a curve (between the curve and the x -axis). Today we extend that idea to find areas trapped between two curves.

1 Area Between Two Curves: The Basic Setup

1.1 From One Curve to Two

Recall: the definite integral $\int_a^b f(x) dx$ gives the (signed) area between $f(x)$ and the x -axis on $[a, b]$.

New situation: We have two continuous functions $f(x)$ and $g(x)$ on $[a, b]$, with $f(x) \geq g(x)$ throughout. We want the area of the region sandwiched between their graphs.

The idea: Slice the region into thin vertical rectangles of width Δx . A typical rectangle at position x_i^* has height $f(x_i^*) - g(x_i^*)$ (top curve minus bottom curve). Summing and taking the limit gives a Riemann sum that converges to a definite integral.

Theorem 1 (Area Between Two Curves). *Let $f(x)$ and $g(x)$ be continuous on $[a, b]$ with $f(x) \geq g(x)$ for all $x \in [a, b]$. Then the area of the region between the graphs is:*

$$A = \int_a^b [f(x) - g(x)] dx$$

Key point: It doesn't matter whether the curves are above or below the x -axis. What matters is which curve is on top. Always subtract the bottom curve from the top curve.

1.2 Worked Examples

Example 1 (Interval Given). Find the area of the region bounded above by $f(x) = x + 4$ and below by $g(x) = 3 - \frac{x}{2}$ on $[1, 4]$.

Solution: Since $f(x) \geq g(x)$ throughout (you can verify at $x = 1$: $5 \geq 2.5$), we compute directly:

$$\begin{aligned} A &= \int_1^4 [(x + 4) - (3 - \frac{x}{2})] dx \\ &= \int_1^4 [\frac{3x}{2} + 1] dx \\ &= \left[\frac{3x^2}{4} + x \right]_1^4 \\ &= (12 + 4) - \left(\frac{3}{4} + 1 \right) \\ &= 16 - \frac{7}{4} = \frac{57}{4} \end{aligned}$$

The area is $\frac{57}{4}$ units².

1.3 Practice Problem

Work this out: Find the area of the region bounded by $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over $[1, 5]$.

Solution: Check which is on top at $x = 1$: $f(1) = 5.5$ and $g(1) = 1.5$, so $f \geq g$ throughout.

$$\begin{aligned} A &= \int_1^5 \left[\left(\frac{x}{2} + 5 \right) - \left(x + \frac{1}{2} \right) \right] dx = \int_1^5 \left[\frac{9}{2} - \frac{x}{2} \right] dx \\ &= \left[\frac{9x}{2} - \frac{x^2}{4} \right]_1^5 = \left(\frac{45}{2} - \frac{25}{4} \right) - \left(\frac{9}{2} - \frac{1}{4} \right) \\ &= \frac{65}{4} - \frac{17}{4} = \frac{48}{4} = 12 \end{aligned}$$

The area is 12 units².

1.4 Finding the Interval from the Intersection Points

Often the problem doesn't give you a and b directly—you find them by setting $f(x) = g(x)$ and solving.

Example 2 (Interval Not Given). Find the area of the region bounded above by $f(x) = 9 - \left(\frac{x}{2}\right)^2$ and below by $g(x) = 6 - x$.

Solution:

Step 1: Find the intersection points. Set $f(x) = g(x)$:

$$\begin{aligned}9 - \frac{x^2}{4} &= 6 - x \\36 - x^2 &= 24 - 4x \\x^2 - 4x - 12 &= 0 \\(x - 6)(x + 2) &= 0\end{aligned}$$

So $x = -2$ and $x = 6$.

Step 2: Identify the top curve. At $x = 0$: $f(0) = 9$ and $g(0) = 6$, so $f \geq g$ on $[-2, 6]$.

Step 3: Integrate.

$$\begin{aligned}A &= \int_{-2}^6 \left[9 - \frac{x^2}{4} - (6 - x) \right] dx \\&= \int_{-2}^6 \left[3 + x - \frac{x^2}{4} \right] dx \\&= \left[3x + \frac{x^2}{2} - \frac{x^3}{12} \right]_{-2}^6 \\&= (18 + 18 - 18) - \left(-6 + 2 + \frac{2}{3} \right) \\&= 18 - \left(-\frac{10}{3} \right) = \frac{64}{3}\end{aligned}$$

The area is $\frac{64}{3}$ units².

1.5 Practice Problem

Work this out: Find the area of the region bounded above by $f(x) = x$ and below by $g(x) = x^4$.

Solution: Find intersections: $x = x^4 \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0$, so $x = 0$ and $x = 1$. On $[0, 1]$, $x \geq x^4$ (check at $x = 1/2$: $1/2 > 1/16$).

$$A = \int_0^1 (x - x^4) dx = \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

The area is $\frac{3}{10}$ units².

2 Areas of Compound Regions (Curves That Cross)

2.1 The Problem

What if the graphs of f and g cross somewhere inside the interval? Then neither function is always on top, and using $\int_a^b [f(x) - g(x)] dx$ would cause cancellation—positive and negative

areas would offset each other.

The fix: Use the absolute value $|f(x) - g(x)|$ to always subtract the smaller from the larger:

Theorem 2 (Area When Curves Cross). *If f and g are continuous on $[a, b]$, the area of the region between their graphs is:*

$$A = \int_a^b |f(x) - g(x)| dx$$

In practice, find all crossing points, split $[a, b]$ into sub-intervals where one curve is consistently on top, and add the resulting integrals.

Example 3 (Crossing Curves). *Find the area between $f(x) = \sin x$ and $g(x) = \cos x$ on $[0, \pi]$.*

Solution:

The graphs intersect at $x = \pi/4$ (set $\sin x = \cos x$, so $\tan x = 1$).

- On $[0, \pi/4]$: $\cos x \geq \sin x$, so $|f - g| = \cos x - \sin x$.
- On $[\pi/4, \pi]$: $\sin x \geq \cos x$, so $|f - g| = \sin x - \cos x$.

$$\begin{aligned} A &= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\ &= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi} \\ &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 \right) + \left(1 - 0 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \right) \\ &= (\sqrt{2} - 1) + (1 + \sqrt{2}) \\ &= 2\sqrt{2} \end{aligned}$$

The area is $2\sqrt{2}$ units².

2.2 Practice Problem

Work this out: Find the area between $f(x) = \sin x$ and $g(x) = \cos x$ on $[\pi/2, 2\pi]$.

Solution: Find crossings on $[\pi/2, 2\pi]$: $\sin x = \cos x \Rightarrow \tan x = 1$, so $x = 5\pi/4$.

- On $[\pi/2, 5\pi/4]$: $\sin x \geq \cos x$ (check at $x = \pi$: $0 \geq -1$). ✓
- On $[5\pi/4, 2\pi]$: $\cos x \geq \sin x$ (check at $x = 3\pi/2$: $0 \geq -1$). ✓

$$\begin{aligned}
A &= \int_{\pi/2}^{5\pi/4} (\sin x - \cos x) dx + \int_{5\pi/4}^{2\pi} (\cos x - \sin x) dx \\
&= [-\cos x - \sin x]_{\pi/2}^{5\pi/4} + [\sin x + \cos x]_{5\pi/4}^{2\pi} \\
&= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 - 1) \right) + \left((0 + 1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \right) \\
&= (\sqrt{2} + 1) + (1 + \sqrt{2}) = 2 + 2\sqrt{2}
\end{aligned}$$

The area is $2 + 2\sqrt{2}$ units².

3 Integrating with Respect to y

3.1 When Switching Variables Helps

Sometimes a region is awkward to describe using vertical slices (integrals in x) but natural using horizontal slices (integrals in y).

Setup: Express the boundary curves as $x = v(y)$ (left curve) and $x = u(y)$ (right curve) with $u(y) \geq v(y)$. Integrate over the y -interval $[c, d]$:

Theorem 3 (Area by Integrating Along the y -Axis). *If $u(y) \geq v(y)$ for all $y \in [c, d]$, the area of the region between the curves is:*

$$A = \int_c^d [u(y) - v(y)] dy$$

The mechanics are the same as before—right curve minus left curve—just with horizontal rectangles of height Δy .

Example 4 (Integrating in y). *Find the area of the region bounded by $x = y^2$ (left) and $x = 2 - y$ (right) by integrating with respect to y .*

Solution:

Step 1: Find intersection. Set $y^2 = 2 - y$:

$$y^2 + y - 2 = 0 \implies (y + 2)(y - 1) = 0$$

So $y = -2$ and $y = 1$.

Step 2: Identify right and left. At $y = 0$: $u(0) = 2 > v(0) = 0$, so $u(y) = 2 - y$ is to the right.

Step 3: Integrate.

$$\begin{aligned} A &= \int_{-2}^1 [(2-y) - y^2] dy \\ &= \left[2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^1 \\ &= \left(2 - \frac{1}{2} - \frac{1}{3} \right) - \left(-4 - 2 + \frac{8}{3} \right) \\ &= \frac{7}{6} - \left(-6 + \frac{8}{3} \right) = \frac{7}{6} + 6 - \frac{8}{3} = \frac{9}{2} \end{aligned}$$

The area is $\frac{9}{2}$ units².

3.2 When Should You Integrate in y ?

Integrating with respect to y is often preferable when:

- The region has a boundary that is a function of y but not easily a function of x (e.g., $x = y^2$).
- Integrating in x would require splitting into two or more integrals, but integrating in y needs only one.

3.3 Practice Problem

Work this out: Find the area of the region bounded by $x = y^2$ and $x = 9$ by integrating with respect to y .

Solution: The curves intersect where $y^2 = 9$, so $y = -3$ and $y = 3$. On $[-3, 3]$, $x = 9$ is to the right of $x = y^2$.

$$\begin{aligned} A &= \int_{-3}^3 (9 - y^2) dy = \left[9y - \frac{y^3}{3} \right]_{-3}^3 \\ &= (27 - 9) - (-27 + 9) = 18 - (-18) = 36 \end{aligned}$$

The area is 36 units².

4 Summary

- **Basic formula (in x):** $A = \int_a^b [f(x) - g(x)] dx$ when $f \geq g$ on $[a, b]$.
- **Finding limits:** If not given, solve $f(x) = g(x)$ to find intersection points.

- **Curves that cross:** Split the interval at each crossing point; always subtract the lower curve from the upper curve on each sub-interval.
- **Integrating in y :** $A = \int_c^d [u(y) - v(y)] dy$ (right minus left). Useful when horizontal slices are simpler.