Determining Volumes by Slicing
Section 6.2
Calculus I - Lecture Notes

April 16, 2025

Motivating Question

A loaf of bread has a curved shape we can't compute with $V = lwh$. Can integration help?

Last class we used integration to find *areas* by slicing a region into thin rectangles. Today we use the same idea one dimension up: we slice a *solid* into thin slabs and integrate to find its *volume*.

1 The Slicing Method

1.1 The Big Idea

Suppose a solid S extends along the x-axis from $x = a$ to $x = b$. At each x, we can look at the **cross-section**—the shape you'd see if you sliced the solid perpendicular to the x-axis at that point.

Let $A(x)$ denote the area of the cross-section at position x. If we slice the solid into n thin slabs of thickness Δx, then the volume of each slab is approximately $A(x_i^*) \cdot \Delta x$. Adding them all up and taking the limit gives:

Theorem 1 (Volume by Slicing)**.** *Let $A(x)$ be the cross-sectional area of a solid S perpendicular to the x-axis at position x. Then the volume of S is:*

$$V = \int_a^b A(x)\, dx$$

This is the **slicing method**. The strategy is:

1. Determine the shape of each cross-section.

2. Find a formula $A(x)$ for the area of that cross-section.

3. Integrate $A(x)$ over the appropriate interval.

Example 1 (Deriving the Volume of a Pyramid)**.** *A pyramid with a square base of side a and height h. Derive $V = \frac{1}{3}a^2h$ using slicing.*

Solution:
Orient the pyramid with its apex at the origin and base at $x = h$. Cross-sections perpendicular to the x-axis are squares. By similar triangles, the side length of the square at position x is $s = \frac{ax}{h}$. So:

$$A(x) = s^2 = \frac{a^2x^2}{h^2}$$

Integrating from 0 to h:

$$V = \int_0^h \frac{a^2x^2}{h^2}\,dx = \frac{a^2}{h^2} \cdot \frac{x^3}{3}\bigg|_0^h = \frac{a^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}a^2h$$

This confirms the formula from geometry.

1.2 Practice Problem

Work this out: A solid has a circular base of radius 2 and square cross-sections perpendicular to the x-axis. Set up (but do not necessarily evaluate) the integral for the volume.

Solution: Place the base circle $x^2 + y^2 = 4$ in the xy-plane. At position $x \in [-2, 2]$, the cross-section is a square whose side equals the full chord of the circle: $s = 2\sqrt{4 - x^2}$. So:

$$A(x) = s^2 = 4(4 - x^2)$$

$$V = \int_{-2}^{2} 4(4 - x^2)\,dx$$

(Evaluating: $V = 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 4 \cdot \frac{32}{3} = \frac{128}{3}$ units^3.)

2 Solids of Revolution: The Disk Method

2.1 What Is a Solid of Revolution?

Take a flat region in the xy-plane and spin it around an axis (like a line $y = c$ or $x = c$). The resulting three-dimensional shape is called a **solid of revolution**. Think of a potter spinning clay on a wheel.

2.2 Cross-Sections Are Disks

When we revolve the region under $y = f(x)$ (above the x-axis) around the x-axis, each cross-section perpendicular to the x-axis is a **disk** (a filled circle). The disk at position x has:

$$\text{radius} = f(x), \qquad A(x) = \pi[f(x)]^2$$

Plugging into the slicing formula gives:

Theorem 2 (The Disk Method—x-axis). *Let $f(x)$ be continuous and nonnegative on $[a, b]$. The volume of the solid obtained by revolving the region under f around the x-axis is:*

$$V = \int_a^b \pi[f(x)]^2\,dx$$

Example 2 (Disk Method, x-axis). *Find the volume of the solid obtained by revolving the region under $f(x) = \sqrt{x}$ on $[1, 4]$ around the x-axis.*

Solution:
Each cross-section is a disk of radius $\sqrt{x}$.

$$V = \int_1^4 \pi(\sqrt{x})^2\,dx = \pi \int_1^4 x\,dx = \pi \cdot \frac{x^2}{2}\bigg|_1^4 = \pi\left(8 - \frac{1}{2}\right) = \frac{15\pi}{2}$$

The volume is $\frac{15\pi}{2}$ units^3.

2.3 Revolving Around the y-Axis

The disk method works just as well around the y-axis. Express the boundary as a function of y and integrate with respect to y:

Theorem 3 (The Disk Method—y-axis). *Let $g(y)$ be continuous and nonnegative on $[c, d]$. The volume of the solid obtained by revolving the region to the left of $x = g(y)$ around the y-axis is:*

$$V = \int_c^d \pi[g(y)]^2\,dy$$

Example 3 (Disk Method, y-axis). *Find the volume obtained by revolving the region bounded by $g(y) = \sqrt{4 - y}$ and the y-axis over $[0, 4]$ around the y-axis.*

Solution:
Each cross-section (perpendicular to the y-axis) is a disk of radius $\sqrt{4 - y}$.

$$V = \int_0^4 \pi(\sqrt{4 - y})^2\,dy = \pi \int_0^4 (4 - y)\,dy = \pi\left[4y - \frac{y^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$$

The volume is 8π units^3.

2.4 Practice Problem

Work this out: Find the volume of the solid obtained by revolving the region under $f(x) = 4 - x$ on $[0, 4]$ around the x-axis.

Solution: Each cross-section is a disk of radius $4 - x$.

$$V = \int_0^4 \pi(4-x)^2\,dx = \pi\int_0^4 (16 - 8x + x^2)\,dx$$
$$= \pi\left[16x - 4x^2 + \frac{x^3}{3}\right]_0^4 = \pi\Big(64 - 64 + \frac{64}{3}\Big) = \frac{64\pi}{3}$$

The volume is $\dfrac{64\pi}{3}$ units^3.

3 Solids of Revolution: The Washer Method

3.1 When There Is a Hole

What if we revolve the region *between* two curves, rather than between a curve and an axis? The resulting solid has a cavity in the middle—like a washer (or a donut cross-section).

At position x, the cross-section is a **washer**: an outer circle of radius $f(x)$ with an inner circle of radius $g(x)$ removed. Its area is:

$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi\big([f(x)]^2 - [g(x)]^2\big)$$

Theorem 4 (The Washer Method—x-axis)**.** *Let $f(x) \geq g(x) \geq 0$ on $[a, b]$. The volume of the solid obtained by revolving the region between f and g around the x-axis is:*

$$V = \int_a^b \pi\big([f(x)]^2 - [g(x)]^2\big)\,dx$$

Memory device: $\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$.

Example 4 (Washer Method)**.** *Find the volume of the solid obtained by revolving the region between $f(x) = x$ and $g(x) = \frac{1}{x}$ on $[1, 4]$ around the x-axis.*

Solution:
On $[1, 4]$, $f(x) = x \geq g(x) = 1/x \geq 0$.

$$V = \int_1^4 \pi\left[x^2 - \frac{1}{x^2}\right]dx = \pi\left[\frac{x^3}{3} + \frac{1}{x}\right]_1^4$$
$$= \pi\left[\left(\frac{64}{3} + \frac{1}{4}\right) - \left(\frac{1}{3} + 1\right)\right]$$
$$= \pi\left[\frac{63}{3} + \frac{1}{4} - 1\right] = \pi\left[21 - \frac{3}{4}\right] = \frac{81\pi}{4}$$

The volume is $\dfrac{81\pi}{4}$ units^3.

3.2 A Different Axis of Revolution

We can also revolve around a horizontal line $y = k$ that is not the x-axis. In this case, the radii of the outer and inner circles shift by k.

Example 5 (Washer Method, Axis $y = -2$)**.** *Find the volume of the solid obtained by revolving the region under $f(x) = 4 - x$ on $[0, 4]$ around the line $y = -2$.*

Solution:
The axis of revolution is $y = -2$, which lies below the region.

- ***Outer radius:*** *distance from $y = -2$ to $y = f(x) = 4 - x$ is $(4 - x) - (-2) = 6 - x$.*
- ***Inner radius:*** *distance from $y = -2$ to $y = 0$ (the x-axis boundary) is $0 - (-2) = 2$.*

$$\begin{aligned} V &= \int_0^4 \pi\left[(6-x)^2 - 2^2\right] dx = \pi \int_0^4 (x^2 - 12x + 32)\, dx \\ &= \pi \left[\frac{x^3}{3} - 6x^2 + 32x\right]_0^4 \\ &= \pi \left(\frac{64}{3} - 96 + 128\right) = \pi \left(\frac{64}{3} + 32\right) = \frac{160\pi}{3} \end{aligned}$$

The volume is $\frac{160\pi}{3}$ units^3.

3.3 Practice Problem

Work this out: Find the volume of the solid obtained by revolving the region between $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$ on $[1, 3]$ around the x-axis.

Solution: On $[1, 3]$, $\sqrt{x} \geq 1/x \geq 0$ (check at $x = 1$: both equal 1; at $x = 2$: $\sqrt{2} \approx 1.41 > 0.5$). Use the washer method with outer radius $\sqrt{x}$ and inner radius $1/x$:

$$\begin{aligned} V &= \int_1^3 \pi \left[(\sqrt{x})^2 - \left(\frac{1}{x}\right)^2\right] dx = \pi \int_1^3 \left[x - \frac{1}{x^2}\right] dx \\ &= \pi \left[\frac{x^2}{2} + \frac{1}{x}\right]_1^3 = \pi \left[\left(\frac{9}{2} + \frac{1}{3}\right) - \left(\frac{1}{2} + 1\right)\right] \\ &= \pi \left[\frac{29}{6} - \frac{3}{2}\right] = \pi \cdot \frac{20}{6} = \frac{10\pi}{3} \end{aligned}$$

The volume is $\frac{10\pi}{3}$ units^3.

4 Disk vs. Washer: When to Use Each

Situation	Method	Formula
Region between $f(x)$ and x-axis	Disk	$\int_a^b \pi[f(x)]^2\,dx$
Region between $f(y)$ and y-axis	Disk	$\int_c^d \pi[g(y)]^2\,dy$
Region between two curves (hole present)	Washer	$\int_a^b \pi\big([f]^2 - [g]^2\big)\,dx$

Note: The disk method is just a special case of the washer method where the inner radius is zero.

5 Summary

- **Slicing method:** $V = \int_a^b A(x)\,dx$, where $A(x)$ is the cross-sectional area at x. Works for any solid, not just solids of revolution.
- **Disk method:** For a solid of revolution around the x-axis with no hole, cross-sections are disks:

$$V = \int_a^b \pi[f(x)]^2\,dx$$

- **Washer method:** For a solid of revolution with a cavity (region between two curves), cross-sections are washers:

$$V = \int_a^b \pi\big([f(x)]^2 - [g(x)]^2\big)\,dx$$

- For a non-standard axis of revolution $y = k$: shift the radii appropriately (outer radius = distance from axis to outer boundary, inner radius = distance from axis to inner boundary).