

# Integration Formulas & Net Change Theorem

## Section 5.4

### Calculus I - Lecture Notes

April 2, 2026

#### Motivating Question

A car drives north at 40 mph for 2 hours, then south at 30 mph for 1 hour.  
Where does it end up?

We can answer this by thinking about displacement: going north gives  $+40 \times 2 = 80$  miles, going south gives  $-30 \times 1 = -30$  miles. Net result:  $80 - 30 = 50$  miles north of the starting point.

If we model velocity as  $v(t) = 40$  on  $[2, 4]$  and  $v(t) = -30$  on  $[4, 5]$  (negative = southward), the definite integral gives us this automatically:

$$\int_2^5 v(t) dt = 80 + (-30) = 50 \text{ miles north of start.}$$

But if someone asks *how far did the car travel in total?*, the answer is  $80 + 30 = 110$  miles—we care about the *absolute* amount of motion, not the direction. Today we make this distinction precise and develop the **Net Change Theorem**.

## Basic Integration Formulas

By FTC Part 2, every antiderivative rule is also an integration rule. The table below collects everything you need. Memorize the first two rows; the trig rules follow the same pattern.

## 0.1 The Integration Formula Table

$$\begin{aligned}\int x^n dx &= \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) & \int \cos x dx &= \sin x + C \\ \int \sin x dx &= -\cos x + C & \int \sec^2 x dx &= \tan x + C \\ \int \csc^2 x dx &= -\cot x + C & \int \sec x \tan x dx &= \sec x + C \\ \int \csc x \cot x dx &= -\csc x + C\end{aligned}$$

Constant multiple and sum rules carry over:

$$\int cf(x) dx = c \int f(x) dx, \quad \int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx.$$

## 1 Applying the Integration Formulas

**Common setup step:** Before integrating, simplify the integrand—distribute, split fractions, rewrite roots as fractional exponents—so the power rule applies directly.

**Example 1.** Use the power rule to evaluate  $\int_1^4 \sqrt{t}(1+t) dt$ .

**Solution:**

First, distribute and rewrite using exponents:

$$\sqrt{t}(1+t) = t^{1/2}(1+t) = t^{1/2} + t^{3/2}.$$

Now integrate using the power rule:

$$\begin{aligned}\int_1^4 (t^{1/2} + t^{3/2}) dt &= \left( \frac{t^{3/2}}{\frac{3}{2}} + \frac{t^{5/2}}{\frac{5}{2}} \right) \Big|_1^4 \\ &= \left( \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} \right) \Big|_1^4 \\ &= \left( \frac{2}{3}(8) + \frac{2}{5}(32) \right) - \left( \frac{2}{3}(1) + \frac{2}{5}(1) \right) \\ &= \left( \frac{16}{3} + \frac{64}{5} \right) - \left( \frac{2}{3} + \frac{2}{5} \right) \\ &= \frac{14}{3} + \frac{62}{5} = \frac{70}{15} + \frac{186}{15} = \frac{256}{15}.\end{aligned}$$

**Example 2.** Evaluate  $\int_0^\pi (\sin x - \cos x) dx$ .

**Solution:**

$$\begin{aligned}\int_0^\pi (\sin x - \cos x) dx &= (-\cos x - \sin x) \Big|_0^\pi \\ &= (-\cos \pi - \sin \pi) - (-\cos 0 - \sin 0) \\ &= (1 - 0) - (-1 - 0) = 1 + 1 = 2.\end{aligned}$$

## 1.1 Practice Problem

Evaluate  $\int_1^2 \left(x^2 - \frac{1}{x^2}\right) dx$ .

**Solution:**

$$\begin{aligned}\int_1^2 (x^2 - x^{-2}) dx &= \left(\frac{x^3}{3} + \frac{1}{x}\right) \Big|_1^2 \\ &= \left(\frac{8}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + 1\right) \\ &= \left(\frac{8}{3} + \frac{1}{2}\right) - \frac{4}{3} \\ &= \frac{4}{3} + \frac{1}{2} = \frac{8}{6} + \frac{3}{6} = \frac{11}{6}.\end{aligned}$$

# The Net Change Theorem

## 2 Statement and Meaning

FTC Part 2 says  $\int_a^b F'(x) dx = F(b) - F(a)$ . Rearranging gives a very useful way to read it:

**Theorem 1** (Net Change Theorem). *Final value = Initial value +  $\int$  (rate of change).*

$$F(b) = F(a) + \int_a^b F'(x) dx.$$

This is the same equation as FTC Part 2, just solved for  $F(b)$ . The power is in the interpretation: whenever you know a *rate* and want the *total change*, integrate. Some examples:

Rate of change $F'(x)$	Quantity $F(x)$	Net change $\int_a^b F' dx$
velocity (ft/s)	position (ft)	displacement (ft)
speed (ft/s)	distance (ft)	distance traveled
marginal cost (\$/unit)	total cost (\$)	change in cost
population growth rate	population	population change

### 3 Displacement vs. Total Distance

**Definition 1.** Let  $v(t)$  be the velocity of an object on  $[t_1, t_2]$ .

- **Net displacement** (where it ended up, can be negative):  $\int_{t_1}^{t_2} v(t) dt$ .
- **Total distance traveled** (always  $\geq 0$ ):  $\int_{t_1}^{t_2} |v(t)| dt$ .

**How to compute total distance, step by step:**

1. Find all times  $t$  in  $[t_1, t_2]$  where  $v(t) = 0$  (the particle changes direction).
2. Split the integral at those times.
3. On each sub-interval, replace  $|v|$  with  $+v$  (if moving forward) or  $-v$  (if moving backward).
4. Add the pieces.

**Example 3** (Net Displacement). A particle has velocity  $v(t) = 3t - 5$  m/s on  $[0, 3]$ . Find the net displacement.

**Solution:**

$$\begin{aligned} \int_0^3 (3t - 5) dt &= \left( \frac{3t^2}{2} - 5t \right) \Big|_0^3 \\ &= \left( \frac{27}{2} - 15 \right) - 0 = \frac{27}{2} - \frac{30}{2} = -\frac{3}{2}. \end{aligned}$$

Net displacement is  $-\frac{3}{2}$  m. The particle ends up 1.5 m behind its starting position.

**Example 4** (Total Distance). For the same particle ( $v(t) = 3t - 5$  on  $[0, 3]$ ), find the total distance traveled.

**Solution:**

Find where  $v(t) = 0$ :  $3t - 5 = 0 \implies t = \frac{5}{3}$ .

The two subintervals are  $\left[0, \frac{5}{3}\right]$  (where  $v < 0$ ) and  $\left[\frac{5}{3}, 3\right]$  (where  $v > 0$ ).

$$\begin{aligned}
\int_0^3 |v(t)| dt &= \int_0^{5/3} (5 - 3t) dt + \int_{5/3}^3 (3t - 5) dt \\
&= \left( 5t - \frac{3t^2}{2} \right) \Big|_0^{5/3} + \left( \frac{3t^2}{2} - 5t \right) \Big|_{5/3}^3 \\
&= \left( \frac{25}{3} - \frac{25}{6} \right) + \left( \frac{27}{2} - 15 \right) - \left( \frac{25}{6} - \frac{25}{3} \right) \\
&= \frac{25}{6} + \left( -\frac{3}{2} \right) - \left( -\frac{25}{6} \right) \\
&= \frac{25}{6} - \frac{9}{6} + \frac{25}{6} = \frac{41}{6} \approx 6.83 \text{ m.}
\end{aligned}$$

### 3.1 Practice Problem

A particle has velocity  $v(t) = t^2 - 3t - 18$  on  $[0, 6]$ . Find (a) the net displacement and (b) the total distance traveled.

**Solution:**

(a) **Net displacement:**

$$\begin{aligned}
\int_0^6 (t^2 - 3t - 18) dt &= \left( \frac{t^3}{3} - \frac{3t^2}{2} - 18t \right) \Big|_0^6 \\
&= (72 - 54 - 108) - 0 = -90 \text{ m.}
\end{aligned}$$

(b) **Total distance:** Factor  $v(t) = (t - 6)(t + 3)$ . On  $[0, 6]$ , the only zero is at  $t = 6$  (and  $t = -3$  is outside the domain). Check:  $v(1) = 1 - 3 - 18 = -20 < 0$ , so  $v \leq 0$  on all of  $[0, 6)$ . Therefore  $|v(t)| = -v(t)$  throughout:

$$\int_0^6 |v(t)| dt = \int_0^6 (-t^2 + 3t + 18) dt = \left( -\frac{t^3}{3} + \frac{3t^2}{2} + 18t \right) \Big|_0^6 = (-72 + 54 + 108) = 90 \text{ m.}$$

The particle moves 90 m in one direction the entire time, ending 90 m from its start. Net displacement is  $-90$  m (displacement is negative because it moved in the negative direction); total distance is 90 m.

## Even and Odd Functions

### 4 Symmetry and Integration

If the limits are symmetric about 0 (i.e.,  $[-a, a]$ ), you can exploit the symmetry of the function to cut the work in half.

**Quick check:** Even functions ( $f(-x) = f(x)$ ) include  $x^2, x^4, \cos x$ , constants. Odd functions ( $f(-x) = -f(x)$ ) include  $x, x^3, \sin x$ .

**Theorem 2** (Integrals of Even and Odd Functions). *Let  $f$  be continuous on  $[-a, a]$ .*

- If  $f$  is **even** ( $f(-x) = f(x)$ ):  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ .
- If  $f$  is **odd** ( $f(-x) = -f(x)$ ):  $\int_{-a}^a f(x) dx = 0$ .

**Why?** Even functions are symmetric about the  $y$ -axis—the area on  $[-a, 0]$  equals the area on  $[0, a]$ , so they add. Odd functions are symmetric about the origin—the signed areas on the two halves cancel.

**Example 5.** Evaluate  $\int_{-2}^2 (3x^8 - 2) dx$  using symmetry.

**Solution:**

$f(x) = 3x^8 - 2$  is even (all terms have even powers of  $x$ , and constants are even). Therefore:

$$\begin{aligned} \int_{-2}^2 (3x^8 - 2) dx &= 2 \int_0^2 (3x^8 - 2) dx \\ &= 2 \left( \frac{x^9}{3} - 2x \right) \Big|_0^2 \\ &= 2 \left( \frac{512}{3} - 4 \right) = 2 \cdot \frac{500}{3} = \frac{1000}{3}. \end{aligned}$$

**Example 6.** Evaluate  $\int_{-\pi}^{\pi} (-5 \sin x) dx$ .

**Solution:**

$f(x) = -5 \sin x$  is odd because  $\sin(-x) = -\sin x$ , so  $f(-x) = -5 \sin(-x) = 5 \sin x = -f(x)$ .

By the odd-function rule on a symmetric interval:

$$\int_{-\pi}^{\pi} (-5 \sin x) dx = 0.$$

This makes geometric sense: the positive lobe on  $[-\pi, 0]$  and negative lobe on  $[0, \pi]$  have equal area and cancel.

## 4.1 Practice Problem

Evaluate  $\int_{-2}^2 x^4 dx$  using symmetry.

**Solution:**

$f(x) = x^4$  is even ( $(-x)^4 = x^4$ ). By the even-function rule:

$$\int_{-2}^2 x^4 dx = 2 \int_0^2 x^4 dx = 2 \cdot \frac{x^5}{5} \Big|_0^2 = 2 \cdot \frac{32}{5} = \frac{64}{5}.$$

## 5 An Applied Example: Iceboat Racing

**Example 7.** Andrew's iceboat travels at twice the wind speed. The wind speed (mph) is

$$v(t) = \begin{cases} 20t + 5 & 0 \leq t \leq \frac{1}{2} \\ 15 & \frac{1}{2} \leq t \leq 1 \end{cases}$$

How far does Andrew travel in 1 hour?

**Solution:**

Andrew's speed is  $2v(t)$ . Integrate over each piece:

$$\begin{aligned} \text{Distance} &= \int_0^1 2v(t) dt = \int_0^{1/2} 2(20t + 5) dt + \int_{1/2}^1 2(15) dt \\ &= \int_0^{1/2} (40t + 10) dt + \int_{1/2}^1 30 dt \\ &= (20t^2 + 10t) \Big|_0^{1/2} + 30t \Big|_{1/2}^1 \\ &= (5 + 5) + (30 - 15) = 10 + 15 = 25 \text{ miles.} \end{aligned}$$

*This is the Net Change Theorem in action: integrate the rate (speed) to get the total change (distance).*

## 6 Summary

**Integration formula table:** Know the power rule, trig integrals

**Net Change Theorem:**

$$\int_a^b F'(x) dx = F(b) - F(a).$$

The integral of a rate of change = net change in quantity.

**Displacement vs. distance:**

- Net displacement =  $\int_{t_1}^{t_2} v(t) dt$  (can be negative).

- Total distance =  $\int_{t_1}^{t_2} |v(t)| dt$  (always  $\geq 0$ ): split at zeros of  $v$ .

**Even/odd shortcut on  $[-a, a]$ :**

- Even function:  $\int_{-a}^a f dx = 2 \int_0^a f dx$ .
- Odd function:  $\int_{-a}^a f dx = 0$ .

**Looking ahead:** Next class (Section 5.5) we introduce  $u$ -substitution—the integration counterpart of the chain rule, and the first of several techniques that dramatically expand which integrals we can evaluate.