Volumes of Revolution: Cylindrical Shells
Section 6.3
Calculus I - Lecture Notes

April 21, 2025

Motivating Question

The disk/washer method integrates *along* the axis of revolution. What if integrating *perpendicular* to it is easier?

Last class we found volumes by slicing a solid into disks or washers. Today we introduce a third method: **cylindrical shells**. Instead of cutting perpendicular to the axis of revolution, we peel the solid like an onion into thin cylindrical layers. This lets us keep our variable of integration perpendicular to the axis, which is often more convenient.

1 The Method of Cylindrical Shells

1.1 Where the Formula Comes From

Consider the region R bounded above by $y = f(x)$, below by the x-axis, and on the sides by $x = a$ and $x = b$. We revolve R around the **y-axis**.

Instead of slicing with horizontal disks (which would require us to express everything in terms of y), we take a thin vertical rectangle at position x of width Δx and height $f(x)$, and revolve it around the y-axis. The rectangle sweeps out a thin **cylindrical shell**.

Volume of one shell: Think of cutting the shell open and flattening it into a rectangular slab.

- Height of slab: $f(x)$
- Width of slab (circumference of cylinder): $2\pi x$
- Thickness: Δx

So $V_{\text{shell}} \approx 2\pi x \cdot f(x) \cdot \Delta x$.

Summing all shells and taking $n \to \infty$:

Theorem 1 (The Shell Method — revolving around the y-axis)**.** *Let $f(x)$ be continuous and nonnegative on $[a, b]$. The volume of the solid formed by revolving the region under f around the y-axis is:*

$$V = \int_a^b 2\pi x\, f(x)\, dx$$

Memory device: $V = \int 2\pi \cdot (\text{radius}) \cdot (\text{height})\, dx$.

1.2 Comparing Shell and Disk/Washer

Feature	**Disk/Washer**	**Shell**
Slices	Perpendicular to axis	Parallel to axis
Typical element	Disk or washer	Thin cylinder
Revolving around y-axis	Integrate in y	Integrate in x
Revolving around x-axis	Integrate in x	Integrate in y

Key insight: with shells, you integrate *perpendicular* to the axis of revolution. So when revolving around the y-axis, you integrate in x — no need to solve for x as a function of y.

1.3 Worked Examples

Example 1 (Basic Shell Method)**.** *Find the volume of the solid formed by revolving the region under $f(x) = 2x - x^2$ on $[0, 2]$ around the y-axis.*

Solution:

$$\begin{aligned} V &= \int_0^2 2\pi x(2x - x^2)\, dx = 2\pi \int_0^2 (2x^2 - x^3)\, dx \\ &= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = 2\pi \left(\frac{16}{3} - 4\right) = 2\pi \cdot \frac{4}{3} = \frac{8\pi}{3} \end{aligned}$$

The volume is $\frac{8\pi}{3}$ *units*^{3}.

1.4 Practice Problem

Work this out: Find the volume of the solid formed by revolving the region under $f(x) = x^2$ on $[1, 2]$ around the y-axis.

Solution:

$$V = \int_1^2 2\pi x \cdot x^2\, dx = 2\pi \int_1^2 x^3\, dx = 2\pi \left[\frac{x^4}{4}\right]_1^2 = 2\pi \left(4 - \frac{1}{4}\right) = \frac{15\pi}{2}$$

The volume is $\frac{15\pi}{2}$ units^3.

2 Shells Around the x-Axis

When revolving around the x-axis instead, we use horizontal shells. Express the boundary as a function of y and integrate in y:

Theorem 2 (The Shell Method — revolving around the x-axis)**.** *Let $g(y)$ be continuous and nonnegative on $[c, d]$. The volume of the solid formed by revolving the region to the left of $x = g(y)$ around the x-axis is:*

$$V = \int_c^d 2\pi y \, g(y) \, dy$$

Example 2 (Shell Method Around the x-Axis)**.** *Find the volume of the solid formed by revolving the region to the right of the y-axis and below $g(y) = 2\sqrt{y}$ on $[0, 4]$ around the x-axis.*

Solution:

$$V = \int_0^4 2\pi y \cdot 2\sqrt{y} \, dy = 4\pi \int_0^4 y^{3/2} \, dy = 4\pi \left[\frac{2y^{5/2}}{5} \right]_0^4 = 4\pi \cdot \frac{64}{5} = \frac{256\pi}{5}$$

The volume is $\frac{256\pi}{5}$ *units*^{3}.

2.1 Practice Problem

Work this out: Find the volume of the solid formed by revolving the region to the right of the y-axis and below $g(y) = 3/y$ on $[1, 3]$ around the x-axis.

Solution:

$$V = \int_1^3 2\pi y \cdot \frac{3}{y} \, dy = \int_1^3 6\pi \, dy = 6\pi \, y \Big|_1^3 = 6\pi(3 - 1) = 12\pi$$

The volume is 12π units^3.

3 Non-Standard Axes and Two-Curve Regions

3.1 Revolving Around $x = k$

When the axis of revolution is $x = k$ rather than the y-axis, the radius of each shell is the distance from the shell at position x to the axis. If the axis is to the left of the region ($k < a$), the radius is $x - k$. More generally:

$$V = \int_a^b 2\pi \, (\text{radius}) \, f(x) \, dx, \qquad \text{where radius} = |x - k|$$

Example 3 (Non-Standard Axis)**.** *Find the volume formed by revolving the region under $f(x) = x$ on $[1, 2]$ around the line $x = -1$.*

Solution:

The axis $x = -1$ is to the left of the region, so the radius of the shell at position x is $x - (-1) = x + 1$.

$$\begin{aligned} V &= \int_1^2 2\pi(x+1)\cdot x\,dx = 2\pi\int_1^2 (x^2+x)\,dx \\ &= 2\pi\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_1^2 = 2\pi\left[\left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+\frac{1}{2}\right)\right] = 2\pi\cdot\frac{23}{6} = \frac{23\pi}{3} \end{aligned}$$

The volume is $\frac{23\pi}{3}$ units^3.

3.2 Region Between Two Curves

When the region is bounded above by $f(x)$ and below by $g(x)$, the height of each shell is $f(x) - g(x)$:

$$V = \int_a^b 2\pi x\big[f(x) - g(x)\big]\,dx$$

Example 4 (Two-Curve Shell)**.** *Find the volume formed by revolving the region between $f(x) = \sqrt{x}$ and $g(x) = 1/x$ on $[1, 4]$ around the y-axis.*

Solution:

On $[1, 4]$, $\sqrt{x} \geq 1/x$, so the shell height is $\sqrt{x} - 1/x$.

$$\begin{aligned} V &= \int_1^4 2\pi x\left(\sqrt{x}-\frac{1}{x}\right)\,dx = 2\pi\int_1^4 \left(x^{3/2}-1\right)\,dx \\ &= 2\pi\left[\frac{2x^{5/2}}{5}-x\right]_1^4 = 2\pi\left[\left(\frac{64}{5}-4\right)-\left(\frac{2}{5}-1\right)\right] \\ &= 2\pi\left[\frac{62}{5}-3\right] = 2\pi\cdot\frac{47}{5} = \frac{94\pi}{5} \end{aligned}$$

The volume is $\frac{94\pi}{5}$ units^3.

3.3 Practice Problem

Work this out: Find the volume formed by revolving the region between $f(x) = x$ and $g(x) = x^2$ on $[0, 1]$ around the y-axis.

Solution:

On $[0, 1]$, $x \geq x^2$.

$$\begin{aligned} V &= \int_0^1 2\pi x(x-x^2)\,dx = 2\pi\int_0^1 (x^2-x^3)\,dx \\ &= 2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{\pi}{6} \end{aligned}$$

The volume is $\frac{\pi}{6}$ units^3.

4 Which Method Should You Use?

When you have a choice, pick the method that avoids the most algebra. Here are the key questions:

1. **Is the region bounded on both sides by the same curve?** Then shells in that direction are problematic — use disks.
2. **Would the disk/washer method require splitting the integral?** If yes, shells may give a single cleaner integral.
3. **Is expressing the boundary as a function of y difficult?** If yes, use shells (integrate in x) when revolving around the y-axis.

Example 5 (Choosing the Best Method)**.** *Region bounded by $y = 4x - x^2$ and the x-axis, revolved around the x-axis.*

Which method? *The region is bounded on both left and right by the same curve $y = 4x - x^2$. Trying to define a horizontal rectangle leads to ambiguity. Shells around the x-axis would require writing $y = 4x - x^2$ as a function of y — messy. Use **disks**:*

$$V = \int_0^4 \pi(4x - x^2)^2 \, dx$$

5 Summary

- **Shell formula (around y-axis):** $V = \displaystyle\int_a^b 2\pi x \, f(x) \, dx$. Integrate in x.
- **Shell formula (around x-axis):** $V = \displaystyle\int_c^d 2\pi y \, g(y) \, dy$. Integrate in y.
- **Non-standard axis $x = k$:** Replace x with the actual radius (distance from shell to axis).
- **Two-curve region:** Replace $f(x)$ with $f(x) - g(x)$ (top minus bottom) as the shell height.
- **Choose shells when** the disk/washer method would require splitting the integral or awkward algebra. Choose disks/washers when the region is bounded on both sides by the same curve.