

# Arc Length of a Curve and Surface Area

## Section 6.4

### Calculus I - Lecture Notes

April 23, 2025

## Motivating Question

We can compute areas and volumes with integration. Can we also measure the *length* of a curved path?

Imagine driving along a winding road. The straight-line distance from start to finish is not the distance you actually travel. Today we use the same Riemann sum idea to build a formula for the **arc length** of a curve, and then extend it to find the **surface area** of a solid of revolution.

## 1 Arc Length of $y = f(x)$

### 1.1 Deriving the Formula

To find the length of the curve  $y = f(x)$  from  $x = a$  to  $x = b$ , we approximate it with  $n$  short line segments connecting consecutive points on the curve.

By the Pythagorean theorem, the length of the  $i$ -th segment connecting  $(x_{i-1}, f(x_{i-1}))$  to  $(x_i, f(x_i))$  is:

$$\ell_i = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \Delta x \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}$$

By the Mean Value Theorem, there exists  $x_i^* \in [x_{i-1}, x_i]$  with  $\frac{\Delta y_i}{\Delta x} = f'(x_i^*)$ , so:

$$\ell_i = \sqrt{1 + [f'(x_i^*)]^2} \Delta x$$

Summing and taking the limit:

**Theorem 1** (Arc Length for  $y = f(x)$ ). *Let  $f(x)$  be smooth (differentiable with continuous derivative) on  $[a, b]$ . Then the arc length of the graph of  $f$  from  $(a, f(a))$  to  $(b, f(b))$  is:*

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

**Note:** The integrand  $\sqrt{1 + [f'(x)]^2}$  is always  $\geq 1$ , which makes sense: arc length is always at least as long as the horizontal distance  $b - a$ .

## 1.2 Worked Examples

**Example 1** (Arc Length, Exact Answer). Find the arc length of  $f(x) = 2x^{3/2}$  on  $[0, 1]$ .

**Solution:**

$$f'(x) = 3x^{1/2}, \text{ so } [f'(x)]^2 = 9x.$$

$$L = \int_0^1 \sqrt{1 + 9x} \, dx$$

Substitute  $u = 1 + 9x$ ,  $du = 9 \, dx$ . When  $x = 0$ ,  $u = 1$ ; when  $x = 1$ ,  $u = 10$ .

$$L = \frac{1}{9} \int_1^{10} \sqrt{u} \, du = \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10} = \frac{2}{27} (10^{3/2} - 1) \approx 2.268 \text{ units}$$

**Example 2** (Arc Length Requiring a Calculator). Find the arc length of  $f(x) = x^2$  on  $[1, 3]$ .

**Solution:**

$$f'(x) = 2x, \text{ so } [f'(x)]^2 = 4x^2.$$

$$L = \int_1^3 \sqrt{1 + 4x^2} \, dx$$

This integral has no elementary antiderivative. Using a calculator:  $L \approx 8.268$  units.

**Key lesson:** Most arc length integrals cannot be evaluated by hand. The formula is conceptually important even when the computation requires numerical methods.

## 1.3 Practice Problem

**Work this out:** Find the arc length of  $f(x) = \frac{4}{3}x^{3/2}$  on  $[0, 1]$ . (This one can be done exactly.)

**Solution:**

$$f'(x) = 2x^{1/2}, \text{ so } [f'(x)]^2 = 4x.$$

$$L = \int_0^1 \sqrt{1 + 4x} \, dx$$

Substitute  $u = 1 + 4x$ ,  $du = 4 \, dx$ . When  $x = 0$ ,  $u = 1$ ; when  $x = 1$ ,  $u = 5$ .

$$L = \frac{1}{4} \int_1^5 \sqrt{u} \, du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^5 = \frac{1}{6} (5^{3/2} - 1) = \frac{5\sqrt{5} - 1}{6} \approx 1.697 \text{ units}$$

## 2 Arc Length of $x = g(y)$

When the curve is given as  $x = g(y)$ , we partition the  $y$ -axis instead. The same derivation gives:

**Theorem 2** (Arc Length for  $x = g(y)$ ). *Let  $g(y)$  be smooth on  $[c, d]$ . Then the arc length of the graph of  $g$  is:*

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} dy$$

**Example 3** (Arc Length as a Function of  $y$ ). *Find the arc length of  $g(y) = 3y^3$  on  $[1, 2]$ .*

**Solution:**

$$g'(y) = 9y^2, \text{ so } [g'(y)]^2 = 81y^4.$$

$$L = \int_1^2 \sqrt{1 + 81y^4} dy \approx 21.028 \text{ units (calculator)}$$

### 2.1 Practice Problem

**Work this out:** Set up (but do not evaluate) the arc length integral for  $g(y) = 1/y$  on  $[1, 4]$ . Then use a calculator to approximate it.

**Solution:**

$$g'(y) = -1/y^2, \text{ so } [g'(y)]^2 = 1/y^4.$$

$$L = \int_1^4 \sqrt{1 + \frac{1}{y^4}} dy \approx 3.132 \text{ units (calculator)}$$

## 3 Surface Area of a Surface of Revolution

### 3.1 The Idea

When we revolve the curve  $y = f(x)$  around the  $x$ -axis, each small arc segment of length  $ds = \sqrt{1 + [f'(x)]^2} dx$  sweeps out a thin band whose area is the circumference  $2\pi f(x)$  times the arc length  $ds$ :

$$dS = 2\pi f(x) ds = 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

Integrating:

**Theorem 3** (Surface Area of a Surface of Revolution). *Let  $f(x)$  be smooth and nonnegative on  $[a, b]$ . The surface area generated by revolving the graph of  $f$  around the  $x$ -axis is:*

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

Similarly, revolving  $x = g(y)$  around the  $y$ -axis gives:

$$S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy$$

**Pattern to remember:**

$$S = \int 2\pi \cdot (\text{radius}) \cdot \sqrt{1 + (\text{derivative})^2} d(\text{variable})$$

The  $\sqrt{1 + (\text{derivative})^2}$  factor is exactly the arc length element  $ds$  — we are multiplying the circumference at each point by the arc length of the curve at that point.

### 3.2 Worked Examples

**Example 4** (Surface Area, Revolving Around the  $x$ -Axis). *Find the surface area generated by revolving  $f(x) = \sqrt{x}$  on  $[1, 4]$  around the  $x$ -axis.*

**Solution:**

$$f'(x) = \frac{1}{2\sqrt{x}}, \text{ so } [f'(x)]^2 = \frac{1}{4x}.$$

$$S = \int_1^4 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}} dx = \int_1^4 2\pi\sqrt{x + \frac{1}{4}} dx$$

Substitute  $u = x + 1/4$ ,  $du = dx$ . When  $x = 1$ ,  $u = 5/4$ ; when  $x = 4$ ,  $u = 17/4$ .

$$S = 2\pi \int_{5/4}^{17/4} \sqrt{u} du = 2\pi \cdot \frac{2}{3} u^{3/2} \Big|_{5/4}^{17/4} = \frac{\pi}{6} [17\sqrt{17} - 5\sqrt{5}] \approx 30.846 \text{ units}^2$$

**Example 5** (Surface Area, Revolving Around the  $y$ -Axis). *Consider  $y = 3x^3$ , i.e.  $x = g(y) = \frac{1}{3}y^{1/3}$ , for  $0 \leq y \leq 2$ . Find the surface area generated by revolving the graph around the  $y$ -axis.*

**Solution:**

$$g'(y) = y^{-2/3}, \text{ so } [g'(y)]^2 = y^{-4}.$$

$$S = \int_0^2 2\pi \cdot \frac{y^{1/3}}{3} \sqrt{1 + y^{-4}} dy = \frac{2\pi}{3} \int_0^2 y^{1/3} \sqrt{1 + y^{-4}} dy$$

Substitute  $u = y^4 + 1$ ,  $du = 4y^3 dy$ .

$$S = \frac{2\pi}{3} \cdot \frac{1}{4} \int_1^{17} \sqrt{u} du = \frac{\pi}{6} \cdot \frac{2}{3} u^{3/2} \Big|_1^{17} = \frac{\pi}{9} [17^{3/2} - 1] \approx 24.118 \text{ units}^2$$

### 3.3 Practice Problem

**Work this out:** Find the surface area generated by revolving  $f(x) = 1 - x$  on  $[0, 1/2]$  around the  $x$ -axis.

**Solution:**

$$f'(x) = -1, \text{ so } [f'(x)]^2 = 1.$$

$$\begin{aligned} S &= \int_0^{1/2} 2\pi(1-x)\sqrt{1+1} dx = 2\sqrt{2}\pi \int_0^{1/2} (1-x) dx \\ &= 2\sqrt{2}\pi \left[ x - \frac{x^2}{2} \right]_0^{1/2} = 2\sqrt{2}\pi \left( \frac{1}{2} - \frac{1}{8} \right) = 2\sqrt{2}\pi \cdot \frac{3}{8} = \frac{3\sqrt{2}\pi}{4} \approx 3.330 \text{ units}^2 \end{aligned}$$

## 4 Arc Length vs. Surface Area: A Quick Comparison

Quantity	Formula	Analogous to
Arc length of $y = f(x)$	$\int_a^b \sqrt{1 + [f']^2} dx$	—
Surface area (rev. around $x$ -axis)	$\int_a^b 2\pi f(x) \sqrt{1 + [f']^2} dx$	$2\pi r \cdot (\text{arc length})$
Arc length of $x = g(y)$	$\int_c^d \sqrt{1 + [g']^2} dy$	—
Surface area (rev. around $y$ -axis)	$\int_c^d 2\pi g(y) \sqrt{1 + [g']^2} dy$	$2\pi r \cdot (\text{arc length})$

## 5 Summary

- **Arc length of  $y = f(x)$ :**  $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$ . Requires  $f$  to be smooth.
- **Arc length of  $x = g(y)$ :**  $L = \int_c^d \sqrt{1 + [g'(y)]^2} dy$ . Same idea, different variable.
- Most arc length integrals cannot be evaluated in closed form and require numerical approximation.
- **Surface area (around  $x$ -axis):**  $S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$ . Think: circumference times arc length element.
- **Surface area (around  $y$ -axis):**  $S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy$ .