Integration by Substitution
Section 5.5
Calculus I - Lecture Notes

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Motivating Question

How do we integrate functions that are too complicated for our basic rules?

The Fundamental Theorem of Calculus tells us that $\int f(x)\,dx = F(x) + C$ whenever $F'(x) = f(x)$. But what if $f(x)$ is something like $(3x^2+4)^{100} \cdot 6x$? We could expand the power — but that's 101 terms.

There is a better way. Recall the **chain rule**:

$$\frac{d}{dx}\big[F(g(x))\big] = F'(g(x)) \cdot g'(x).$$

If we integrate both sides, we get

$$\int F'(g(x)) \cdot g'(x)\,dx = F(g(x)) + C.$$

Integration by substitution (also called u-substitution or change of variables) is exactly the chain rule run backwards. We look for an integrand of the form $f(g(x)) \cdot g'(x)$ and replace the complicated inner piece with a single new variable u.

1 The Substitution Rule

Theorem 1 (Substitution with Indefinite Integrals)**.** *Let $u = g(x)$, where $g'(x)$ is continuous, and let f be continuous on the range of g. Then*

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du = F(u) + C = F\big(g(x)\big) + C.$$

1.1 The Substitution Idea in One Picture

The key step is recognizing which piece to call u. We write $du = g'(x)\,dx$ (the "differential") and check that everything in the integrand – including dx – converts cleanly to u's and du's.

Problem-Solving Strategy:

1. Identify a piece $g(x)$ inside the integrand whose derivative $g'(x)$ also appears.
2. Set $u = g(x)$ and compute $du = g'(x)\,dx$.
3. Rewrite the entire integrand in terms of u and du.
4. Integrate with respect to u.
5. Substitute back to get the answer in terms of x.

2 Indefinite Integrals via Substitution

2.1 Straightforward Match

Example 1. *Find* $\int 6x\left(3x^2+4\right)^4 dx$.

Solution:
Step 1: *Set* $u = 3x^2 + 4$.
Step 2: *Differentiate:* $du = 6x\,dx$. *The factor* $6x\,dx$ *appears exactly in the integrand.*
Step 3: *Rewrite:*

$$\int 6x\left(3x^2+4\right)^4 dx = \int u^4\,du.$$

Step 4: *Integrate:*

$$\int u^4\,du = \frac{u^5}{5} + C.$$

Step 5: *Substitute back:*

$$= \frac{(3x^2+4)^5}{5} + C.$$

Check: *Differentiate* $\frac{(3x^2+4)^5}{5}$. *By the chain rule:*

$$\frac{d}{dx}\left[\frac{(3x^2+4)^5}{5}\right] = \frac{5(3x^2+4)^4 \cdot 6x}{5} = 6x(3x^2+4)^4.\checkmark$$

2.2 Practice Problem

Work this out: Find $\int 3x^2\left(x^3-3\right)^2 dx$.

2.3 Adjusting for a Missing Constant Factor

Sometimes $g'(x)$ is *almost* there but off by a constant. We can fix this by multiplying du by a compensating constant.

Example 2. *Find* $\int z\sqrt{z^2-5}\,dz$.

Solution:
Rewrite the integrand as $\int z(z^2-5)^{1/2}\,dz$.
Step 1: *Set* $u = z^2 - 5$.
Step 2: $du = 2z\,dz$, *so* $z\,dz = \frac{1}{2}\,du$.
Step 3: *Rewrite (pull the* $\frac{1}{2}$ *outside):*

$$\int z(z^2-5)^{1/2}\,dz = \frac{1}{2}\int u^{1/2}\,du.$$

Step 4: *Integrate:*

$$\frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \frac{1}{2}\cdot\frac{2}{3}\,u^{3/2} + C = \frac{1}{3}u^{3/2} + C.$$

Step 5: *Substitute back:*

$$= \frac{1}{3}(z^2-5)^{3/2} + C.$$

2.4 Practice Problem

Work this out: Find $\int x^2\left(x^3+5\right)^9\,dx$.

2.5 Trigonometric Integrands

Substitution works just as well when the inner function involves trig.

Example 3. *Find* $\int \frac{\sin t}{\cos^3 t}\,dt$.

Solution:
Step 1: *Set* $u = \cos t$ *(the inner piece being raised to a power).*
Step 2: $du = -\sin t\,dt$, *so* $\sin t\,dt = -du$.
Step 3: *Rewrite:*

$$\int \frac{\sin t}{\cos^3 t}\,dt = \int \frac{-du}{u^3} = -\int u^{-3}\,du.$$

Step 4: *Integrate:*

$$-\int u^{-3}\,du = -\cdot\frac{u^{-2}}{-2} + C = \frac{1}{2u^2} + C.$$

Step 5: *Substitute back:*

$$= \frac{1}{2\cos^2 t} + C.$$

2.6 Practice Problem

Work this out: Find $\displaystyle\int \frac{\cos t}{\sin^2 t}\,dt$.

2.7 When You Have to Solve for x in Terms of u

Sometimes a factor of the original variable x remains after substitution and is not part of du. In this case, use the substitution equation itself to express x in terms of u.

Example 4. *Find* $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$.

Solution:
Step 1: *Set* $u = x - 1$.
Step 2: $du = dx$. *But the numerator has* x, *not* $x - 1$. *Solve:* $x = u + 1$.
Step 3: *Rewrite:*

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \left(u^{1/2} + u^{-1/2}\right) du.$$

Step 4: *Integrate term by term:*

$$\int \left(u^{1/2} + u^{-1/2}\right) du = \frac{2}{3}u^{3/2} + 2u^{1/2} + C.$$

Step 5: *Substitute back and factor:*

$$\begin{aligned}
&= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C\\
&= (x-1)^{1/2}\left[\frac{2}{3}(x-1) + 2\right] + C\\
&= (x-1)^{1/2}\left[\frac{2x-2+6}{3}\right] + C\\
&= \frac{2}{3}(x-1)^{1/2}(x+2) + C.
\end{aligned}$$

3 Definite Integrals via Substitution

When the limits of integration are numbers, we have two options:

- Find the antiderivative in terms of x first, then evaluate at the original limits.
- **(Preferred)** Change the limits to u-values at the same time as the substitution and evaluate directly.

Theorem 2 (Substitution with Definite Integrals)**.** *If $u = g(x)$ and g' is continuous on $[a, b]$, and f is continuous on the range of g, then*

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Key point: When you substitute $u = g(x)$, replace the limits $x = a$ and $x = b$ with $u = g(a)$ and $u = g(b)$. Do *not* convert back to x at the end.

Example 5. *Evaluate* $\int_0^1 x^2(1+2x^3)^5\,dx$*.*

Solution:
Step 1: *Let* $u = 1 + 2x^3$*.*
Step 2: $du = 6x^2\,dx$*, so* $x^2\,dx = \frac{1}{6}\,du$*.*
Step 3 (change limits):

$$x = 0 \Rightarrow u = 1 + 2(0)^3 = 1,$$
$$x = 1 \Rightarrow u = 1 + 2(1)^3 = 3.$$

Step 4: *Rewrite and integrate:*

$$\int_0^1 x^2(1+2x^3)^5\,dx = \frac{1}{6}\int_1^3 u^5\,du = \frac{1}{6}\cdot\frac{u^6}{6}\bigg|_1^3 = \frac{1}{36}\big(3^6 - 1^6\big) = \frac{729-1}{36} = \frac{728}{36} = \frac{182}{9}.$$

3.1 Practice Problem

Work this out: Evaluate $\int_{-1}^0 y(2y^2-3)^5\,dy$.

Example 6. *Evaluate* $\int_0^{\pi/2} \cos^2\theta\,d\theta$*.*

Solution:

We cannot substitute directly. Instead, use the trig identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

to rewrite the integral:

$$\int_0^{\pi/2} \cos^2\theta \, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta \, d\theta.$$

The first integral needs no substitution:

$$\frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

For the second integral, let $u = 2\theta$, so $du = 2\,d\theta$, i.e. $d\theta = \frac{1}{2}\,du$. New limits: $\theta = 0 \Rightarrow u = 0$; $\theta = \pi/2 \Rightarrow u = \pi$.

$$\frac{1}{2}\int_0^{\pi/2} \cos 2\theta \, d\theta = \frac{1}{2}\cdot\frac{1}{2}\int_0^{\pi} \cos u \, du = \frac{1}{4}\sin u\Big|_0^{\pi} = \frac{1}{4}(\sin\pi - \sin 0) = 0.$$

Therefore:

$$\int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{\pi}{4} + 0 = \frac{\pi}{4}.$$

Takeaway: *Sometimes you need a trig identity* before *substitution becomes applicable.*

4 Common Patterns to Recognize

Integrand form	**Let**	**Result**
$\big(g(x)\big)^n \cdot g'(x)$	$u = g(x)$	$\frac{u^{n+1}}{n+1} + C$
$\sin(g(x)) \cdot g'(x)$	$u = g(x)$	$-\cos u + C$
$\cos(g(x)) \cdot g'(x)$	$u = g(x)$	$\sin u + C$

5 Summary

Big idea: Substitution reverses the chain rule. If you see a composite function $f(g(x))$ multiplied by (something close to) $g'(x)$, substitute $u = g(x)$.

Indefinite integrals:

1. Pick $u = g(x)$ so that du appears in the integrand.
2. Adjust for any missing constant (multiply/divide du by a number).

3. Integrate in u, then substitute back.

Definite integrals:

1. Same steps as above, but also convert the limits using $u = g(x)$.
2. Evaluate directly in u without converting back to x.

Watch out for:

- A leftover factor of the original variable (e.g. x not absorbed into du): solve for x in terms of u.
- Trig integrals that need an identity before substitution works ($\cos^2\theta$, $\sin^2\theta$, etc.).
- Missing factors that are *not* just a constant: those cannot be fixed by multiplication; choose a different u.