Functions I

Sections 1.1 & 1.2

Calculus I — Lecture Notes

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Why Calculus?

Big idea: to understand a complicated system, understand how it *changes*.

Most interesting things in the world are complicated and constantly changing: a falling ball, a moving car, a growing bank account. We usually cannot write down the thing itself. But we *can* often say how it is changing at each instant — and calculus is the machine that turns that local information into answers about the whole system. Here are three problems we will be able to solve this semester. None of them can be done with algebra alone.

Example 1: Drop a ball off the Washington Monument

Stand at the top of the Washington Monument (about 176.4 m tall) and drop a ball. **How long until it hits the ground?**

Your first instinct is

$$\text{distance} = \text{speed} \times \text{time}.$$

But *what speed?* The ball leaves your hand at 0 m/s and is moving fast by the bottom — its speed changes every instant. There is no single number to plug in. Algebra is stuck. **That gap is the whole reason calculus exists.**

Here is what we *do* know: how the system changes. Gravity is a constant acceleration

$$a = -9.8 \text{ m/s}^2$$

(negative = downward). Calculus lets us work *backward* from this:

$$\underbrace{a = -9.8}_{\text{acceleration}} \longrightarrow \underbrace{v(t) = -9.8\,t}_{\text{velocity}} \longrightarrow \underbrace{s(t) = -4.9\,t^2}_{\text{distance fallen}}.$$

Notice we never needed the speed — only the rate at which it changes. The ball hits the ground once it has fallen 176.4 m:

$$4.9\,t^2 = 176.4 \quad\Longrightarrow\quad t^2 = 36 \quad\Longrightarrow\quad t = 6 \text{ s}.$$

It slams down at $|v| = 9.8 \times 6 = 58.8$ m/s (about 130 mph).

Example 2: Compute $\sqrt{101}$ in your head

Nobody can do square roots by hand — except you, by the end of today's idea. You *do* know $\sqrt{100} = 10$. The question is how much $\sqrt{x}$ changes when x nudges from 100 to 101.

Calculus says the rate of change of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$, which at $x = 100$ equals $\dfrac{1}{20} = 0.05$. So bumping x up by 1 bumps the square root up by about 0.05:

$$\sqrt{101} \approx 10 + 0.05 = 10.05.$$

A calculator gives $10.0499\ldots$ — we were off by one ten-thousandth, and we did it in our heads. **Knowing how fast a quantity changes lets you predict nearby values.**

Example 3: From your speedometer to your distance

You are driving, and your speed changes every second — there is never one single speed for the trip. Yet your car (and your phone) knows exactly how far you have gone. How?

Calculus runs Example 1 in reverse: it *adds up* a continuously changing speed, instant by instant, to recover the total distance traveled. Same idea as the falling ball, going the other direction.

A rough map of the course

Much of what we do this semester comes back to two questions:

- **How fast is it changing?** (the *derivative*)
- **How much has piled up?** (the *integral*)

One of the nicer results we will see is that these two questions turn out to be closely related — roughly, inverses of each other. But before we can talk about how things change, we need the objects that change — **functions**. That is where we start today.

What Is a Function?

Definition 1. A **function** f is a rule that assigns to each input exactly one output. The set of allowed inputs is the **domain**; the set of outputs is the **range**.

We write $y = f(x)$, read "y equals f of x." Here x is the **independent variable** (the input) and y is the **dependent variable** (the output, since it depends on x). Think of a function as an input/output machine: feed in x, get out $f(x)$. The one rule that makes it a function: *one input cannot produce two different outputs.*

When a function is given by a formula with no stated domain, the domain is the set of all real inputs for which the formula gives a real number (the *natural domain*). For instance $f(x) = x^2$ has domain all of $\mathbb{R}$, while $f(x) = \sqrt{x}$ has domain $[0, \infty)$, since we cannot take the square root of a negative number.

Example 1 (Evaluating a function)**.** For $f(x) = 3x^2 + 2x - 1$, evaluate $f(-2)$ and $f(a + h)$.

Solution. Substitute the input into the formula.

$$f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7,$$

$$f(a + h) = 3(a + h)^2 + 2(a + h) - 1 = 3a^2 + 6ah + 3h^2 + 2a + 2h - 1.$$

(That second one looks pointless now — but $f(a + h)$ is exactly what the derivative is built from. Remember it.)

Example 2 (Domain and range)**.** Find the domain and range of $f(x) = \sqrt{3x + 2} - 1$.

Solution. The square root needs $3x + 2 \geq 0$, i.e. $x \geq -\frac{2}{3}$, so the domain is $\left\{x \mid x \geq -\frac{2}{3}\right\}$. Since $\sqrt{3x + 2} \geq 0$, we have $f(x) \geq -1$, and every value $y \geq -1$ is achieved. The range is $\{y \mid y \geq -1\}$.

Practice (work this out). For $f(x) = x^2 - 3x + 5$, find $f(1)$ and $f(a + h)$.

Combining and composing functions

Given two functions f and g, we can build new functions by arithmetic:

$$(f+g)(x) = f(x)+g(x), \quad (f-g)(x) = f(x)-g(x), \quad (fg)(x) = f(x)g(x), \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\ (g(x) \neq 0).$$

A different, more important way to combine functions is **composition**: feed the output of one function into another.

Definition 2. The **composite function** $g \circ f$ is defined by

$$(g \circ f)(x) = g\big(f(x)\big),$$

with domain all x in the domain of f for which $f(x)$ lies in the domain of g.

Example 3 (Composition)**.** Let $f(x) = x^2 + 1$ and $g(x) = 3x + 1$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

Solution.

$$(g \circ f)(x) = g(x^2 + 1) = 3(x^2 + 1) + 1 = 3x^2 + 4,$$
$$(f \circ g)(x) = f(3x + 1) = (3x + 1)^2 + 1 = 9x^2 + 6x + 2.$$

These are different: **order matters**, $g \circ f \neq f \circ g$.

Composition is not an exotic trick — it is everywhere. When a store takes 20% off ($f(x) = 0.80x$) and then a coupon takes another 15% off the sale price ($g(y) = 0.85y$), the final price is the composition

$$g\big(f(x)\big) = 0.85(0.80x) = 0.68x.$$

It will come back in a big way when we hit the *chain rule*.

Practice (work this out). Let $f(x) = 2 - 5x$ and $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$ and its domain.

Representing Functions

A function can be described four ways — in **words**, by a **table**, by a **graph**, or by a **formula**. The same function can wear all four outfits, and a big part of calculus is moving fluently between them.

The **graph** of f is the set of all points $(x, f(x))$. Two features we read straight off a graph:

- The **zeros** of f: the inputs where $f(x) = 0$ (where the graph crosses the x-axis).
- The **y-intercept**: the point $(0, f(0))$, if 0 is in the domain.

How do we know a curve even *is* a function? Since one input gives only one output, no vertical line can hit the graph twice.

Definition 3 (Vertical Line Test)**.** A set of points in the plane is the graph of a function exactly when no vertical line crosses it more than once.

We also care about where a function rises and falls.

Definition 4. A function f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I, and **decreasing** on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

A function does not need a formula at all. A **table of values** is a perfectly good function — e.g. recording the outside temperature once an hour gives temperature as a function of time, even with no equation in sight. Graphing that table is often how we first *see* the behavior.

Example 4 (Zeros, intercept, and a sketch from a formula)**.** For $f(x) = -4x + 2$, find the zeros and the y-intercept, then sketch.

Solution. Set $f(x) = 0$: $-4x + 2 = 0 \Rightarrow x = \frac{1}{2}$, so the only zero is $\left(\frac{1}{2}, 0\right)$. The y-intercept is $(0, f(0)) = (0, 2)$. Since f is linear, plot those two points and draw the line through them; it falls left to right because the slope -4 is negative.

Piecewise Functions

Some functions use different formulas on different parts of their domain; these are **piecewise-defined**.

Example 5 (A piecewise function)**.** Sketch

$$f(x) = \begin{cases} x + 3, & x < 1, \\ (x-2)^2, & x \geq 1. \end{cases}$$

Solution. Graph the line $y = x + 3$ for $x < 1$ and the parabola $y = (x-2)^2$ for $x \geq 1$. At the switch point $x = 1$: the rule for $x \geq 1$ applies, so $f(1) = (1-2)^2 = 1$ — draw a *closed* dot at $(1, 1)$. The line would have given $1 + 3 = 4$, so draw an *open* dot at $(1, 4)$ to show that value is *not* taken.

The most important piecewise function is the **absolute value**:

$$|x| = \begin{cases} -x, & x < 0, \\ x, & x \geq 0. \end{cases}$$

Its graph is a "V" with the corner at the origin. That corner will matter a lot later: it is the first place we will meet a function that is continuous but has no well-defined slope.

Symmetry: Even and Odd Functions

Definition 5. A function f is **even** if $f(-x) = f(x)$ for all x (graph symmetric about the y-axis). It is **odd** if $f(-x) = -f(x)$ for all x (graph symmetric about the origin).

To test symmetry, plug in $-x$ and compare to $f(x)$ and $-f(x)$.

Example 6 (Even, odd, or neither)**.** Classify each function.

(a) $f(x) = -5x^4 + 7x^2 - 2$

(b) $f(x) = 2x^5 - 4x + 5$

(c) $f(x) = \dfrac{3x}{x^2 + 1}$

Solution.

(a) $f(-x) = -5x^4 + 7x^2 - 2 = f(x)$. **Even.**

(b) $f(-x) = -2x^5 + 4x + 5$, which equals neither $f(x)$ nor $-f(x)$. **Neither.**

(c) $f(-x) = \dfrac{-3x}{x^2 + 1} = -f(x)$. **Odd.**

Practice (work this out). Determine whether $f(x) = 4x^3 - 5x$ is even, odd, or neither.

The Basic Classes of Functions

Now we meet the standard cast of functions we will differentiate and integrate all semester. (We save exponential, logarithmic, and trigonometric functions for the next lecture and beyond.)

Linear functions and slope

The simplest function is **linear**: $f(x) = mx + b$, whose graph is a line. The **slope** m measures how much y changes per unit change in x:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.$$

Two forms worth memorizing:

$$\text{slope-intercept: } y = mx + b, \qquad \text{point-slope: } y - y_1 = m(x - x_1).$$

Example 7 (Equation of a line)**.** Find the line through $(11, -4)$ and $(-4, 5)$.

Solution. The slope is

$$m = \frac{5 - (-4)}{-4 - 11} = \frac{9}{-15} = -\frac{3}{5}.$$

Point-slope through $(11, -4)$: $y + 4 = -\frac{3}{5}(x - 11)$. Solving for y gives slope-intercept form $y = -\frac{3}{5}x + \frac{13}{5}$.

Polynomials

A **polynomial** has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0.$$

The **degree** is n and a_n is the **leading coefficient**. Degree 1 is linear, degree 2 is **quadratic** ($ax^2 + bx + c$), degree 3 is **cubic**. The leading term controls the *end behavior* (what happens as $x \to \pm\infty$).

To find where a polynomial crosses the x-axis, solve $f(x) = 0$. For a quadratic, if it does not factor nicely, use the **quadratic formula**:

$$ax^2 + bx + c = 0 \quad \Longrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 8 (Zeros of a cubic)**.** Find the zeros of $f(x) = x^3 - 3x^2 - 4x$.

Solution. Factor out x, then factor the quadratic:

$$f(x) = x(x^2 - 3x - 4) = x(x - 4)(x + 1).$$

The zeros are $x = 0,\ 4,\ -1$. Since the leading term is x^3 (positive, odd degree), $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

Example 9 (When it doesn't factor: the quadratic formula)**.** Find the zeros of $f(x) = -2x^2 + 4x - 1$ and describe its shape.

Solution. With $a = -2$, $b = 4$, $c = -1$:

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-1)}}{2(-2)} = \frac{-4 \pm \sqrt{8}}{-4} = \frac{2 \pm \sqrt{2}}{2}.$$

So $x \approx 0.29$ and $x \approx 1.71$. The quantity $b^2 - 4ac = 8 > 0$ (the **discriminant**) confirms two real zeros. Because $a = -2 < 0$, the parabola opens *downward.*

Power, root, rational, and algebraic functions

- A **power function** has the form $f(x) = x^b$. When $b = n$ is a positive integer it is a polynomial; when $b = 1/n$ it is a **root function** $f(x) = \sqrt[n]{x}$.
- A **rational function** is a quotient of polynomials, $f(x) = \dfrac{p(x)}{q(x)}$. Its domain excludes the inputs where the denominator is zero.
- An **algebraic function** is anything built from these using $+, -, \times, \div$, powers, and roots — for example $f(x) = \sqrt{4 - x^2}$.

A useful fact about root functions: the domain of $f(x) = \sqrt[n]{x}$ depends on n. For *even* n (like $\sqrt{x}$) the domain is $[0, \infty)$, since we cannot take an even root of a negative number; for *odd* n (like $\sqrt[3]{x}$) the domain is all of $\mathbb{R}$.

Example 10 (Domains of algebraic functions)**.** Find the domain of each function.

(a) $f(x) = \dfrac{3}{x^2 - 1}$
(b) $f(x) = \sqrt{4 - 3x}$
(c) $f(x) = \sqrt[3]{2x - 1}$

Solution.

(a) We cannot divide by zero, and $x^2 - 1 = 0$ at $x = \pm 1$. Domain: $\{x \mid x \neq \pm 1\}$.
(b) An even root needs $4 - 3x \geq 0$, i.e. $x \leq \frac{4}{3}$. Domain: $\left\{x \mid x \leq \frac{4}{3}\right\}$.
(c) A cube root is defined for every real number. Domain: $(-\infty, \infty)$.

Example 11 (Domain of an algebraic function)**.** Find the domain of $f(x) = \sqrt{4 - x^2}$.

Solution. We need $4 - x^2 \geq 0$, i.e. $x^2 \leq 4$, i.e. $-2 \leq x \leq 2$. The domain is $[-2, 2]$.

Practice (work this out). Find all zeros of $f(x) = x^3 - 5x^2 + 6x$, and find the domain of $g(x) = \sqrt{4 - 2x} + 5$.

Summary

- Calculus exists to handle quantities that change continuously: know how a system *changes* and you can recover the system itself (falling ball, $\sqrt{101}$, speedometer).
- The whole course is two inverse questions: the **derivative** (rate of change) and the **integral** (accumulation).
- A **function** assigns one output to each input; know its domain, range, graph, zeros, and where it increases/decreases.
- Tools today: evaluating $f(a+h)$, the vertical line test, piecewise and absolute-value functions, even/odd symmetry.
- The **basic classes**: linear (slope!), polynomial (degree, zeros, end behavior), power/root, rational, and algebraic functions.
- Next time: transformations and the trigonometric functions.