Functions II: Transformations and Trigonometry

Sections 1.2 & 1.3

Calculus I — Lecture Notes

August 27, 2026

Learn a handful of shapes, then learn to move them around — and you know a thousand graphs.

Last time we built the basic cast of functions and learned to read a graph. Today we do two things. First, we learn to **transform** a graph we already know — slide it, stretch it, flip it — so that one familiar shape gives us a whole family of graphs for free. Second, we meet the **trigonometric functions**, the tools for anything that repeats: waves, orbits, springs, sound, the tides. Trig is the language of *cyclical* change, and it will be with us the rest of the semester.

Transformations of Functions

We have already secretly seen a transformation. The graph of $f(x) = (x-2)^2$ is just the parabola $y = x^2$ slid two units to the right. Rather than plotting points all over again, we recognized a known shape and *moved* it. That is the whole game: take a graph you know, and shift, stretch, or reflect it into the one you want.

Shifts (translations)

A **vertical shift** adds a constant to the *output*. For $c > 0$:

$$y = f(x) + c \;\text{ moves the graph } \textit{up } c \text{ units,} \qquad y = f(x) - c \;\text{ moves it } \textit{down } c \text{ units.}$$

A **horizontal shift** adds a constant to the *input*, and here the direction surprises people. For $c > 0$:

$$y = f(x+c) \;\text{ moves the graph } \textit{left } c \text{ units,} \qquad y = f(x-c) \;\text{ moves it } \textit{right } c \text{ units.}$$

Why does $+c$ go *left*? Because the input hits its target value sooner. Take $f(x) = |x+3|$: it bottoms out where $x + 3 = 0$, i.e. at $x = -3$ — three units to the *left* of the corner of $y = |x|$. Inputs move the graph the opposite way from what the sign suggests.

Stretches and compressions (scalings)

Multiplying the *output* by a constant $c > 0$ scales the graph vertically:

$$y = c\,f(x): \quad \text{stretch by a factor of } c \text{ if } c > 1, \qquad \text{compress if } 0 < c < 1.$$

Multiplying the *input* by $c > 0$ scales horizontally — and again the effect is backward from what you would guess:

$$y = f(cx): \quad \text{compress by a factor of } c \text{ if } c > 1, \qquad \text{stretch if } 0 < c < 1.$$

Reflections

Multiplying by -1 flips the graph across an axis:

$y = -f(x)$ reflects across the x-axis (outputs flip sign), $\quad y = f(-x)$ reflects across the y-axis (inputs flip sign)

Here is the full catalog in one place.

Definition 1 (Transformations of $y = f(x)$, with $c > 0$)**.**

$f(x) + c$	shift up c
$f(x) - c$	shift down c
$f(x + c)$	shift left c
$f(x - c)$	shift right c
$c\,f(x)$	vertical stretch if $c > 1$, compression if $0 < c < 1$
$f(cx)$	horizontal compression if $c > 1$, stretch if $0 < c < 1$
$-f(x)$	reflect across the x-axis
$f(-x)$	reflect across the y-axis

When several transformations appear at once, **order matters**. For a combination like $y = c\,f\big(a(x+b)\big) + d$, apply them in this order: (1) horizontal shift, (2) horizontal scaling/reflection, (3) vertical scaling/reflection, (4) vertical shift.

Example 1 (Reading off the transformations)**.** Describe how to obtain the graph of $f(x) = -|x + 2| - 3$ from $y = |x|$.

Solution. Read the operations from the inside out.

- $|x + 2|$: the input has $+2$, so shift $y = |x|$ *left* 2 units.
- $-|x + 2|$: the leading minus sign reflects across the x-axis (the "V" now opens downward).
- $-|x + 2| - 3$: the -3 shifts the whole graph *down* 3 units.

The result is an upside-down "V" with its peak at $(-2, -3)$.

Example 2 (A stretch, a reflection, and a shift)**.** Describe how to obtain $f(x) = 3\sqrt{-x} + 1$ from $y = \sqrt{x}$.

Solution. Again, inside to outside.

- $\sqrt{-x}$: the input is negated, so reflect $y = \sqrt{x}$ across the y-axis. The curve now lives over $x \le 0$.
- $3\sqrt{-x}$: the factor of 3 stretches vertically by a factor of 3.
- $3\sqrt{-x} + 1$: the $+1$ shifts everything *up* 1 unit.

The starting point of the curve moves from the origin up to $(0, 1)$.

Practice (work this out). Describe, in order, the transformations that turn $y = x^2$ into $f(x) = 2(x - 4)^2 - 5$, and state the location of the vertex.

Trigonometric Functions

Almost anything that repeats — a sound wave, alternating current, a swinging pendulum, hours of daylight through the year — can be built out of sines and cosines. To get there we first need a cleaner way to measure angles.

Radian measure

Degrees are an arbitrary human choice (360 for a full turn, because the Babylonians liked 60). **Radians** measure an angle by the arc it cuts on a circle of radius 1.

Definition 2. The **radian measure** of an angle θ is the length s of the arc it subtends on the **unit circle** (the circle of radius 1 centered at the origin). An arc of length 1 corresponds to an angle of 1 radian.

A full circle has circumference 2π, so a full turn is 2π radians; a half turn is π radians. That gives the one conversion fact you need:

$$\boxed{180^\circ = \pi \text{ radians}}, \qquad \text{so} \qquad 1 = \frac{\pi \text{ rad}}{180^\circ} = \frac{180^\circ}{\pi \text{ rad}}.$$

Multiply by whichever copy of 1 cancels the units you want to lose.

Example 3 (Converting between degrees and radians)**.** (a) Express 225° in radians. (b) Express $\frac{5\pi}{3}$ rad in degrees.

Solution. Multiply by the conversion factor that cancels the unwanted unit.

$$\text{(a)} \quad 225^\circ = 225^\circ \cdot \frac{\pi \text{ rad}}{180^\circ} = \frac{225\pi}{180} \text{ rad} = \frac{5\pi}{4} \text{ rad},$$
$$\text{(b)} \quad \frac{5\pi}{3} \text{ rad} = \frac{5\pi}{3} \cdot \frac{180^\circ}{\pi \text{ rad}} = \frac{5 \cdot 180^\circ}{3} = 300^\circ.$$

A few common angles are worth memorizing cold:

$$30^\circ = \frac{\pi}{6}, \quad 45^\circ = \frac{\pi}{4}, \quad 60^\circ = \frac{\pi}{3}, \quad 90^\circ = \frac{\pi}{2}, \quad 180^\circ = \pi, \quad 360^\circ = 2\pi.$$

The six trigonometric functions

Place an angle θ in **standard position**: vertex at the origin, initial side along the positive x-axis, and let the terminal side meet the unit circle at the point $P = (x, y)$. The six trig functions are defined directly from those coordinates.

Definition 3. Let $P = (x, y)$ be the point where the terminal side of θ meets the unit circle. Then

$$\sin\theta = y, \qquad \cos\theta = x, \qquad \tan\theta = \frac{y}{x},$$
$$\csc\theta = \frac{1}{y}, \qquad \sec\theta = \frac{1}{x}, \qquad \cot\theta = \frac{x}{y}.$$

If $x = 0$, then $\sec\theta$ and $\tan\theta$ are undefined; if $y = 0$, then $\csc\theta$ and $\cot\theta$ are undefined.

So $\cos\theta$ is just the x-coordinate and $\sin\theta$ is the y-coordinate of a point traveling around the unit circle — everything else is a ratio built from those two. On a circle of radius r (not just 1), the same angle lands at $(r\cos\theta, r\sin\theta)$.

For an acute angle inside a right triangle, these definitions become the familiar ratios, with O the opposite leg, A the adjacent leg, and H the hypotenuse:

$$\sin\theta = \frac{O}{H}, \qquad \cos\theta = \frac{A}{H}, \qquad \tan\theta = \frac{O}{A} \qquad (\text{“SOH-CAH-TOA”}).$$

The sine and cosine of the first-quadrant reference angles are the backbone of every computation:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Example 4 (Evaluating with the unit circle)**.** Evaluate (a) $\sin\left(\dfrac{2\pi}{3}\right)$ and (b) $\cos\left(-\dfrac{5\pi}{6}\right)$.

Solution.

(a) The angle $\frac{2\pi}{3} = 120°$ sits in the second quadrant, where its terminal side meets the unit circle at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Since sine is the y-coordinate, $\sin\left(\frac{2\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$.

(b) The angle $-\frac{5\pi}{6} = -150°$ is measured clockwise into the third quadrant, with terminal point $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Since cosine is the x-coordinate, $\cos\left(-\frac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$.

Example 5 (Trig in a right triangle: a wooden ramp)**.** A ramp runs from the ground to the top of a staircase 4 ft high, meeting the ground at an angle of θ with $\sin\theta = \frac{1}{2}$ (that is, $\theta = 30°$). How long is the ramp?

Solution. Let x be the ramp length (the hypotenuse). The height 4 is the side opposite θ, so

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{x} \quad\Longrightarrow\quad x = \frac{4}{\sin\theta} = \frac{4}{1/2} = 8 \text{ ft}.$$

The ramp is 8 ft long. (For a shallower $10°$ angle, the same setup gives $x = 4/\sin 10° \approx 23.0$ ft — gentler slope, much longer ramp.)

Practice (work this out). A point $P = \left(\frac{7}{25}, y\right)$ with $y > 0$ lies on the unit circle and is the terminal point of an angle θ. Find y, then compute $\sin\theta$, $\cos\theta$, and $\tan\theta$.

Trigonometric Identities

An **identity** is an equation that holds for *every* angle where both sides are defined. These are the algebraic glue of trigonometry. The essential ones:

$$\text{Reciprocal:}\quad \tan\theta = \frac{\sin\theta}{\cos\theta},\quad \cot\theta = \frac{\cos\theta}{\sin\theta},\quad \csc\theta = \frac{1}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta}.$$

$$\text{Pythagorean:}\quad \sin^2\theta + \cos^2\theta = 1,\qquad 1 + \tan^2\theta = \sec^2\theta,\qquad 1 + \cot^2\theta = \csc^2\theta.$$

$$\text{Double-angle:}\quad \sin(2\theta) = 2\sin\theta\cos\theta,\qquad \cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

The first Pythagorean identity is just the equation of the unit circle ($x^2 + y^2 = 1$) rewritten with $x = \cos\theta$ and $y = \sin\theta$ — so it is not something new to memorize, it is geometry you already know.

Example 6 (Proving an identity)**.** Show that $1 + \tan^2\theta = \sec^2\theta$.

Solution. Start from the identity we trust, $\sin^2\theta + \cos^2\theta = 1$, and divide both sides by $\cos^2\theta$:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}.$$

Since $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ and $\dfrac{1}{\cos\theta} = \sec\theta$, this reads $\tan^2\theta + 1 = \sec^2\theta$, as claimed.

Example 7 (Solving a trig equation)**.** Find all solutions of $1+\cos(2\theta)=\cos\theta$.

Solution. Use the double-angle form $\cos(2\theta)=2\cos^2\theta-1$:

$$1+\big(2\cos^2\theta-1\big)=\cos\theta \quad\Longrightarrow\quad 2\cos^2\theta-\cos\theta=0.$$

Factor — do not divide by $\cos\theta$, or you would lose the solutions where $\cos\theta=0$:

$$\cos\theta\,(2\cos\theta-1)=0.$$

So either $\cos\theta=0$, giving $\theta=\frac{\pi}{2}+n\pi$, or $\cos\theta=\frac{1}{2}$, giving $\theta=\pm\frac{\pi}{3}+2n\pi$, for every integer n.

Practice (work this out). Use a Pythagorean identity to prove that $1+\cot^2\theta=\csc^2\theta$.

Graphs and Sinusoidal Transformations

As a point travels around the unit circle, its coordinates repeat every full turn — so the trig functions are **periodic**. The **period** is the smallest positive p with $f(x+p)=f(x)$ for all x.

- $\sin x$, $\cos x$, $\csc x$, $\sec x$ all have period 2π.
- $\tan x$ and $\cot x$ repeat twice as fast: period π.

The graphs of $y=\sin x$ and $y=\cos x$ are the same wave, offset by $\frac{\pi}{2}$: shifting sine left by $\frac{\pi}{2}$ lands exactly on cosine, so $\cos x=\sin\!\big(x+\frac{\pi}{2}\big)$.

Now we combine trig with the transformations from the start of class. Every wave we care about has the form

$$\boxed{f(x)=A\sin\big(B(x-\alpha)\big)+C}.$$

Each constant is one transformation of the plain sine curve:

- $|A|$ is the **amplitude** — a vertical stretch; the wave rises $|A|$ above and below its center line.
- B sets the **period**, which becomes $\dfrac{2\pi}{|B|}$ — a horizontal compression when $|B|>1$.
- α is the **phase shift** — a horizontal slide of α units.
- C is a **vertical shift** of the center line up to $y=C$.

Example 8 (Reading off amplitude, period, and phase shift)**.** Identify the amplitude, period, and phase shift of $f(x)=3\sin\!\big(2\left(x-\frac{\pi}{4}\right)\big)+1$, and describe the graph.

Solution. Match against $A\sin\big(B(x-\alpha)\big)+C$: here $A=3$, $B=2$, $\alpha=\frac{\pi}{4}$, $C=1$. Therefore

$$\text{amplitude}=|A|=3,\qquad \text{period}=\frac{2\pi}{|B|}=\frac{2\pi}{2}=\pi,\qquad \text{phase shift}=\frac{\pi}{4}\ \text{to the right.}$$

So: take $y=\sin x$, compress horizontally by a factor of 2 (period π), stretch vertically by 3, slide right $\frac{\pi}{4}$, and lift up 1. The wave oscillates between $y=1-3=-2$ and $y=1+3=4$, centered on the line $y=1$.

Example 9 (Building a model: hours of daylight)**.** A city's June 21 (day 172) has 15.7 hours of daylight, its longest; December 21 has 8.3 hours, its shortest. A sine model for daylight hours h as a function of the day of the year t is

$$h(t)=3.7\sin\!\left(\frac{2\pi}{365}\,(t-80.5)\right)+12.$$

Where does each number come from?

Solution. The center line is the average of the extremes, $C = \frac{15.7+8.3}{2} = 12$ hours. The amplitude is half their difference, $A = \frac{15.7-8.3}{2} = 3.7$ hours. The cycle repeats once per year, so $B = \frac{2\pi}{365}$ gives period 365 days. The phase shift $t = 80.5$ (about March 21, the spring equinox) is where the rising wave crosses its center line — exactly a quarter-period before the June peak. Every constant in the model is a physical fact about the year.

Practice (work this out). For $f(x) = 4\cos\big(2\left(x - \frac{\pi}{2}\right)\big)$, state the amplitude, the period, and the phase shift with direction.

Summary

- **Transformations** turn one known graph into a family: $f(x) \pm c$ shifts vertically, $f(x \pm c)$ shifts horizontally (opposite the sign), $c\,f(x)$ and $f(cx)$ scale, and the minus signs in $-f(x)$, $f(-x)$ reflect.
- **Radians** measure angles by arc length on the unit circle; $180^\circ = \pi$ rad is the only conversion you need.
- The six trig functions come straight from the unit circle: $\cos\theta = x$, $\sin\theta = y$, and the rest are ratios; know the $\pi/6,\ \pi/4,\ \pi/3$ values cold.
- Key **identities**: $\sin^2\theta + \cos^2\theta = 1$ (and its two cousins), the reciprocal identities, and the double-angle formulas. When solving, *factor* — never divide by something that could be zero.
- Trig functions are **periodic** (2π for sin/cos, π for tan). A general wave $A\sin\big(B(x-\alpha)\big) + C$ packages amplitude $|A|$, period $2\pi/|B|$, phase shift α, and vertical shift C into one formula.
- Next time: we start asking how these functions *change* — the idea of a limit.