Derivatives as Rates of Change
Section 3.4
Calculus I - Lecture Notes

February 10, 2026

Motivating Question

What can derivatives tell us about the real world?

We've learned how to compute derivatives efficiently. Now we explore what derivatives *mean* in various contexts: physics (motion), biology (population growth), and economics (profit and revenue).

1 Estimating Change with Derivatives

1.1 The Amount of Change Formula

If we know the value of a function at a point and its rate of change, we can estimate nearby values.

Key Idea: For small h, the derivative approximates the average rate of change:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

Solving for $f(a+h)$:

Theorem 1 (Amount of Change Formula)**.**

$$f(a+h) \approx f(a) + f'(a) \cdot h$$

In words: *New value ≈ Old value + (Rate of change)(Change in x)*

This is like using the tangent line to approximate function values!

Example 1. *If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.*

Solution:
Here $a = 3$ and $h = 3.2 - 3 = 0.2$.
Using the formula:

$$\begin{aligned} f(3.2) &= f(3 + 0.2) \\ &\approx f(3) + f'(3) \cdot (0.2) \\ &= 2 + 5(0.2) \\ &= 2 + 1 = 3 \end{aligned}$$

So $f(3.2) \approx 3$.

1.2 Practice Problem

Work this out: Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

2 Motion Along a Line

One of the most important applications of derivatives is analyzing motion.

Definition 1 (Position, Velocity, Speed, and Acceleration). *Let $s(t)$ be the position of an object at time t.*
Velocity: $v(t) = s'(t)$ *(rate of change of position)*
Speed: $|v(t)|$ *(magnitude of velocity, always positive)*
Acceleration: $a(t) = v'(t) = s''(t)$ *(rate of change of velocity)*

Important distinctions:

- Velocity can be positive (moving right/up) or negative (moving left/down)
- Speed is always non-negative
- Acceleration tells us if velocity is increasing or decreasing

Example 2. *A ball is dropped from 64 feet. Its height is $s(t) = -16t^2 + 64$.*
(a) What is the instantaneous velocity when it hits the ground?
(b) What is the average velocity during its fall?

Solution:
First, find when it hits the ground: $-16t^2 + 64 = 0 \Rightarrow t = 2$ seconds.
(a) Find velocity: $v(t) = s'(t) = -32t$
At $t = 2$: $v(2) = -32(2) = -64$ ft/s

(The negative indicates downward motion)
(b) Average velocity:

$$v_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{0 - 64}{2} = -32 \text{ ft/s}$$

2.1 Understanding Velocity and Acceleration Together

Example 3. *A particle's position is $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$, then answer:*
(a) Is the particle moving left or right at $t = 1$?
(b) Is the particle speeding up or slowing down at $t = 1$?

Solution:
Find $v(t) = s'(t) = 3t^2 - 4$ and $a(t) = v'(t) = 6t$.
At $t = 1$: $v(1) = 3(1)^2 - 4 = -1$ and $a(1) = 6(1) = 6$.
(a) Since $v(1) = -1 < 0$, the particle is moving from ***right to left****.*
(b) We have $v(1) < 0$ (moving left) and $a(1) > 0$ (accelerating right).
Velocity and acceleration have ***opposite signs****, so the particle is* ***slowing down****.*
Key insight:

- *If v and a have the* same sign*: speeding up*
- *If v and a have* opposite signs*: slowing down*

Example 4. *A particle's position is $s(t) = t^3 - 9t^2 + 24t + 4$ for $t \geq 0$.*
(a) Find $v(t)$.
(b) When is the particle at rest?
(c) When is the particle moving left? Moving right?

Solution:
(a) $v(t) = s'(t) = 3t^2 - 18t + 24$
(b) Particle is at rest when $v(t) = 0$:

$$\begin{aligned} 3t^2 - 18t + 24 &= 0 \\ 3(t^2 - 6t + 8) &= 0 \\ 3(t - 2)(t - 4) &= 0 \\ t = 2 &\text{ or } t = 4 \end{aligned}$$

(c) Analyze the sign of $v(t) = 3(t - 2)(t - 4)$:
For** $t \in [0, 2)$**:** Both factors negative, so $v(t) > 0 \Rightarrow$ moving* ***right
For** $t \in (2, 4)$**:** $(t - 2) > 0$, $(t - 4) < 0$, so $v(t) < 0 \Rightarrow$ moving* ***left
For** $t \in (4, \infty)$**:** Both factors positive, so $v(t) > 0 \Rightarrow$ moving* ***right

2.2 Practice Problem

Work this out: A particle's position is $s(t) = t^2 - 5t + 1$. Is it moving right or left at $t = 3$?

3 Population Growth

Derivatives model how populations change over time.

Definition 2 (Population Growth Rate)**.** *If $P(t)$ is the number of entities in a population, then the **population growth rate** is $P'(t)$.*

Example 5. *A city's population triples every 5 years. If the current population is 10,000, estimate the population in 2 years.*

Solution:
Let $P(t)$ be the population (in thousands) t years from now.
Given: $P(0) = 10$ and $P(5) = 30$ (triples in 5 years).
Estimate the current growth rate:

$$P'(0) \approx \frac{P(5) - P(0)}{5 - 0} = \frac{30 - 10}{5} = 4 \text{ thousand/year}$$

Using the amount of change formula:

$$P(2) \approx P(0) + P'(0) \cdot 2 = 10 + 4(2) = 18$$

*The population in 2 years will be approximately **18,000**.*

3.1 Practice Problem

Work this out: A mosquito colony has 3,000 mosquitoes. If $P(0) = 3000$ and $P'(0) = 100$ mosquitoes/day, estimate the population in 3 days.

4 Marginal Analysis in Economics

In economics, "marginal" means "the derivative of."

Definition 3 (Marginal Functions)**.** *Let x be the number of items produced/sold.*
Marginal Cost: *$MC(x) = C'(x)$ where $C(x)$ is the cost function*
Marginal Revenue: *$MR(x) = R'(x)$ where $R(x)$ is the revenue function*
Marginal Profit: *$MP(x) = P'(x)$ where $P(x) = R(x) - C(x)$*
Note: $MP(x) = MR(x) - MC(x)$

Interpretation: $C'(x) \approx C(x + 1) - C(x)$
So the marginal cost approximates the cost of producing *one more item.*

Example 6. *The price and demand for barbeque dinners are related by* $p(x) = 9 - 0.03x$ *for* $0 \leq x \leq 300$.

The revenue is $R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x$.
Use marginal revenue to estimate the revenue from selling the 101st dinner.

Solution:
Find marginal revenue: $MR(x) = R'(x) = -0.06x + 9$
At $x = 100$*:* $MR(100) = -0.06(100) + 9 = -6 + 9 = 3$
*This means selling the 101st dinner brings in approximately **$3** in revenue.*
Check: *Actual revenue from 101st dinner:*

$$\begin{aligned} R(101) - R(100) &= [-0.03(101)^2 + 9(101)] - [-0.03(100)^2 + 9(100)] \\ &= 602.97 - 600 = 2.97 \end{aligned}$$

Our estimate of $3 is very close to the actual $2.97!

Example 7. *The profit from selling x fish-fry dinners is* $P(x) = -0.03x^2 + 8x - 50$.
Estimate the profit from selling the 101st dinner.

Solution:
Find marginal profit: $MP(x) = P'(x) = -0.06x + 8$
At $x = 100$*:* $MP(100) = -0.06(100) + 8 = -6 + 8 = 2$
*Selling the 101st dinner adds approximately **$2** to the profit.*

4.1 Practice Problem

Work this out: If the cost function is $C(x) = 200 + 7x + \frac{x^2}{7}$, find the marginal cost $MC(x)$. What does $MC(12)$ represent?

5 Summary of Key Concepts

General Principle: The derivative measures rate of change in any context.

Context	**Function**	**Derivative Means**
Motion	$s(t)$ position	$s'(t)$ = velocity
	$v(t)$ velocity	$v'(t)$ = acceleration
Population	$P(t)$ population	$P'(t)$ = growth rate
Economics	$C(x)$ cost	$C'(x)$ = marginal cost
	$R(x)$ revenue	$R'(x)$ = marginal revenue
	$P(x)$ profit	$P'(x)$ = marginal profit

Amount of Change Formula: $f(a+h) \approx f(a) + f'(a) \cdot h$
Motion Analysis:

- $v(t) > 0$: moving right/up; $v(t) < 0$: moving left/down
- v and a same sign: speeding up
- v and a opposite signs: slowing down
- Particle at rest when $v(t) = 0$

Economic Interpretation:

- Marginal cost $\approx$ cost of one more item
- Marginal revenue $\approx$ revenue from one more sale
- Maximize profit where $MR(x) = MC(x)$