Derivatives of Trigonometric Functions
Section 3.5
Calculus I - Lecture Notes

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Motivating Question

How do we find derivatives of sine, cosine, and other trig functions?

Trigonometric functions model periodic phenomena like oscillations, waves, and circular motion. To analyze these systems (finding velocity, acceleration, rates of change), we need their derivatives.

1 Derivatives of Sine and Cosine

1.1 The Two Fundamental Formulas

These are the building blocks for all other trig derivatives.

Theorem 1 (Derivatives of Sine and Cosine)**.**

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Key observations:

- The derivative of sine is cosine (positive)
- The derivative of cosine is negative sine (note the minus sign!)
- Where $\sin x$ has horizontal tangents (max/min), $\cos x = 0$
- Where $\sin x$ is increasing, $\cos x > 0$; where decreasing, $\cos x < 0$

Proof idea for $\frac{d}{dx}(\sin x) = \cos x$:
We need two key limits (which we proved earlier):

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

Using the definition and the identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right] \\
&= \sin x \cdot 0 + \cos x \cdot 1 = \cos x
\end{aligned}$$

1.2 Using the Basic Formulas

Example 1. *Find the derivative of $f(x) = 5x^3 \sin x$.*

Solution:
Use the product rule:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(5x^3) \cdot \sin x + 5x^3 \cdot \frac{d}{dx}(\sin x) \\
&= 15x^2 \sin x + 5x^3 \cos x
\end{aligned}$$

Example 2. *Find the derivative of $g(x) = \dfrac{\cos x}{4x^2}$.*

Solution:
Use the quotient rule:

$$\begin{aligned}
g'(x) &= \frac{(-\sin x)(4x^2) - (\cos x)(8x)}{(4x^2)^2} \\
&= \frac{-4x^2 \sin x - 8x \cos x}{16x^4} \\
&= \frac{-x \sin x - 2 \cos x}{4x^3}
\end{aligned}$$

1.3 Practice Problem

Work this out: Find the derivative of $f(x) = \sin x \cos x$.

1.4 Application: Finding When a Particle Is at Rest

Example 3. *A particle's position is* $s(t) = 2\sin t - t$ *for* $0 \leq t \leq 2\pi$*. When is the particle at rest?*

Solution:
The particle is at rest when $v(t) = s'(t) = 0$.
Find the derivative: $s'(t) = 2\cos t - 1$
Set equal to zero:

$$\begin{aligned} 2\cos t - 1 &= 0 \\ \cos t &= \frac{1}{2} \\ t &= \frac{\pi}{3} \textit{ or } t = \frac{5\pi}{3} \end{aligned}$$

The particle is at rest at $t = \frac{\pi}{3}$ *and* $t = \frac{5\pi}{3}$.

2 Derivatives of Other Trig Functions

The remaining four trig functions can be expressed using sine and cosine, so we can find their derivatives using quotient rule.

2.1 Derivative of Tangent

Example 4. *Find the derivative of* $f(x) = \tan x$.

Solution:
Write $\tan x = \frac{\sin x}{\cos x}$ *and use quotient rule:*

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$

2.2 All Six Trig Derivatives

Theorem 2 (Derivatives of All Trigonometric Functions)**.**

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Pattern to notice: The co-functions (cosine, cotangent, cosecant) have **negative** derivatives.

Memory aids:

- $\frac{d}{dx}(\tan x) = \sec^2 x$ (think: "tangent to secant squared")
- $\frac{d}{dx}(\sec x) = \sec x \tan x$ (secant times tangent)
- The "co-" versions get minus signs

2.3 Using the Formulas

Example 5. *Find the equation of the tangent line to $f(x) = \cot x$ at $x = \dfrac{\pi}{4}$.*

Solution:

Step 1: *Find the point.*

$$f\left(\frac{\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right) = 1$$

Point: $\left(\frac{\pi}{4}, 1\right)$

Step 2: *Find the slope.*

$$f'(x) = -\csc^2 x, \quad \textit{so} \quad f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -(\sqrt{2})^2 = -2$$

Step 3: *Write the equation.*

$$y - 1 = -2\left(x - \frac{\pi}{4}\right) \quad \textit{or} \quad y = -2x + 1 + \frac{\pi}{2}$$

Example 6. *Find the derivative of $f(x) = \csc x + x \tan x$.*

Solution:

Use sum rule and product rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\csc x) + \frac{d}{dx}(x \tan x) \\ &= -\csc x \cot x + [(1)(\tan x) + (x)(\sec^2 x)] \\ &= -\csc x \cot x + \tan x + x \sec^2 x \end{aligned}$$

2.4 Practice Problem

Work this out: Find the derivative of $f(x) = 2\tan x - 3\cot x$.

3 Higher-Order Derivatives

The derivatives of sine and cosine follow a repeating pattern.

Example 7. *Find the first four derivatives of $y = \sin x$.*

Solution:

$$\begin{aligned} y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x \\ \frac{d^3y}{dx^3} &= -\cos x \\ \frac{d^4y}{dx^4} &= \sin x \end{aligned}$$

The pattern repeats every 4 derivatives!

3.1 Finding Any Higher-Order Derivative

Pattern for $\sin x$:

$$\begin{aligned} \frac{d^1}{dx^1}(\sin x) &= \cos x \\ \frac{d^2}{dx^2}(\sin x) &= -\sin x \\ \frac{d^3}{dx^3}(\sin x) &= -\cos x \\ \frac{d^4}{dx^4}(\sin x) &= \sin x \quad \text{(back to start)} \end{aligned}$$

To find $\frac{d^n}{dx^n}(\sin x)$: Divide n by 4 and look at the remainder.

Example 8. *Find $\dfrac{d^{74}}{dx^{74}}(\sin x)$.*

Solution:
Divide: $74 = 4 \cdot 18 + 2$
The remainder is 2, so:

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x$$

3.2 Practice Problem

Work this out: Find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Hint: What is $59 \div 4$*? What's the remainder?*

4 Application: Simple Harmonic Motion

When objects oscillate (springs, pendulums, waves), their motion is often modeled by trig functions.

Example 9. *A particle moves with position* $s(t) = 2 - \sin t$*. Find* $v\left(\frac{\pi}{4}\right)$ *and* $a\left(\frac{\pi}{4}\right)$*. Is the particle speeding up or slowing down?*

Solution:
Velocity: $v(t) = s'(t) = -\cos t$

$$v\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} < 0$$

Acceleration: $a(t) = v'(t) = \sin t$

$$a\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} > 0$$

Since $v < 0$ *(moving left) and* $a > 0$ *(accelerating right), they have* ***opposite signs****.*
The particle is ***slowing down****.*

4.1 Practice Problem

Work this out: A block on a spring has position $s(t) = 2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Is it speeding up or slowing down?

5 Summary

The Six Trig Derivatives (MEMORIZE):

Function	**Derivative**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Key patterns:

- Co-functions get negative signs
- Use product/quotient rules as needed
- Derivatives of $\sin x$ and $\cos x$ cycle every 4 derivatives

Applications:

- Simple harmonic motion: $s(t) = A\sin(\omega t)$ or $s(t) = A\cos(\omega t)$
- Velocity: $v(t) = s'(t)$
- Acceleration: $a(t) = s''(t)$
- Speeding up when v and a have same sign; slowing down when opposite signs