The Chain Rule
Section 3.6
Calculus I - Lecture Notes

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Motivating Question

How do we differentiate composite functions like $\sin(x^3)$ or $(x^2+1)^{10}$?

We know how to differentiate $\sin x$ and x^3 separately, but what about $\sin(x^3)$? We need a new rule: the **Chain Rule**.

1 The Chain Rule

1.1 The Main Idea

Think of $h(x) = \sin(x^3)$ as a chain reaction:

- As x changes, x^3 changes
- As x^3 changes, $\sin(x^3)$ changes

The rate of change of h depends on *both* rates of change!

Theorem 1 (The Chain Rule)**.** *If $h(x) = f(g(x))$, then*

$$h'(x) = f'(g(x)) \cdot g'(x)$$

In words: *The derivative of a composite function is the derivative of the* outer function *(evaluated at the inner function) times the derivative of the* inner function.

Leibniz notation: If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

1.2 Proof Using Limits

Proof: Let $h(x) = f(g(x))$. By the limit definition:

$$\begin{aligned} h'(x) &= \lim_{h\to 0} \frac{h(x+h) - h(x)}{h} \\ &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \end{aligned}$$

Key trick: Multiply and divide by $g(x+h) - g(x)$:

$$h'(x) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

Split into two limits:

$$h'(x) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

The second limit is $g'(x)$.

For the first limit, let $u = g(x+h) - g(x)$. Since g is continuous, as $h \to 0$, we have $u \to 0$:

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} = \lim_{u\to 0} \frac{f(g(x)+u) - f(g(x))}{u} = f'(g(x))$$

Therefore: $h'(x) = f'(g(x)) \cdot g'(x)$ □

1.3 Understanding the Pattern

For $h(x) = f(g(x))$:

1. Identify the **outer function** f and **inner function** g
2. Take derivative of outer (leaving inner alone)
3. Multiply by derivative of inner

2 The Chain Rule with Power Functions

The most common use: $(g(x))^n$

Theorem 2 (Power Rule with Chain Rule)**.** *If* $h(x) = [g(x)]^n$*, then*

$$h'(x) = n[g(x)]^{n-1} \cdot g'(x)$$

Example 1. *Find the derivative of $f(x) = (x^2+1)^{10}$.*

Solution:
Outer function: u^{10}, Inner function: $u = x^2+1$

$$\begin{aligned} f'(x) &= 10(x^2+1)^9 \cdot \frac{d}{dx}(x^2+1) \\ &= 10(x^2+1)^9 \cdot 2x \\ &= 20x(x^2+1)^9 \end{aligned}$$

Example 2. *Find the derivative of $f(x) = \sqrt{3x-2}$.*

Solution:
Rewrite as $f(x) = (3x-2)^{1/2}$

$$\begin{aligned} f'(x) &= \frac{1}{2}(3x-2)^{-1/2} \cdot 3 \\ &= \frac{3}{2\sqrt{3x-2}} \end{aligned}$$

2.1 Practice Problem

Work this out: Find the derivative of $f(x) = (x^3 - x^2 + 3)^5$.

3 Chain Rule with Trigonometric Functions

Theorem 3 (Chain Rule for Trig Functions)**.**

$$\begin{aligned} \frac{d}{dx}[\sin(g(x))] &= \cos(g(x)) \cdot g'(x) \\ \frac{d}{dx}[\cos(g(x))] &= -\sin(g(x)) \cdot g'(x) \\ \frac{d}{dx}[\tan(g(x))] &= \sec^2(g(x)) \cdot g'(x) \end{aligned}$$

Example 3. *Find the derivative of $f(x) = \cos(x^2)$.*

Solution:

$$\begin{aligned} f'(x) &= -\sin(x^2) \cdot \frac{d}{dx}(x^2) \\ &= -\sin(x^2) \cdot 2x \\ &= -2x\sin(x^2) \end{aligned}$$

Example 4. *Find the derivative of* $f(x) = \sin(x^2 + 3x)$.

Solution:

$$\begin{aligned} f'(x) &= \cos(x^2 + 3x) \cdot \frac{d}{dx}(x^2 + 3x) \\ &= \cos(x^2 + 3x) \cdot (2x + 3) \\ &= (2x + 3)\cos(x^2 + 3x) \end{aligned}$$

3.1 Practice Problem

Work this out: Find the derivative of $f(x) = \tan(x^3)$.

4 Combining Chain Rule with Other Rules

Example 5 (Chain Rule with Product Rule)**.** *Find the derivative of* $f(x) = (3x-1)^4(x^2+1)^7$.

Solution:
Use product rule first, then chain rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[(3x-1)^4] \cdot (x^2+1)^7 + (3x-1)^4 \cdot \frac{d}{dx}[(x^2+1)^7] \\ &= [4(3x-1)^3 \cdot 3](x^2+1)^7 + (3x-1)^4[7(x^2+1)^6 \cdot 2x] \\ &= 12(3x-1)^3(x^2+1)^7 + 14x(3x-1)^4(x^2+1)^6 \end{aligned}$$

Can factor: $= 2(3x-1)^3(x^2+1)^6[6(x^2+1) + 7x(3x-1)]$

Example 6 (Finding a Tangent Line)**.** *Find the equation of the tangent line to* $f(x) = \sqrt{x^2+5}$ *at* $x = 2$.

Solution:
Point: $f(2) = \sqrt{4+5} = 3$, *so* $(2, 3)$
Slope: $f'(x) = \frac{1}{2}(x^2+5)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+5}}$
At $x = 2$*:* $f'(2) = \frac{2}{\sqrt{9}} = \frac{2}{3}$
Equation: $y - 3 = \frac{2}{3}(x-2)$ *or* $y = \frac{2}{3}x + \frac{5}{3}$

5 Composites of Three or More Functions

For $h(x) = f(g(k(x)))$, apply chain rule multiple times:

$$h'(x) = f'(g(k(x))) \cdot g'(k(x)) \cdot k'(x)$$

Work from **outside to inside**, one layer at a time.

Example 7. *Find the derivative of* $h(x) = \cos^3(5x^2 - 1)$.

Solution:
Think: $h(x) = [\cos(5x^2 - 1)]^3$
Layer 1 (outermost): u^3 *where* $u = \cos(5x^2 - 1)$

$$3[\cos(5x^2 - 1)]^2$$

Layer 2: $\cos(v)$ *where* $v = 5x^2 - 1$

$$3[\cos(5x^2 - 1)]^2 \cdot (-\sin(5x^2 - 1))$$

Layer 3 (innermost): $5x^2 - 1$

$$3[\cos(5x^2 - 1)]^2 \cdot (-\sin(5x^2 - 1)) \cdot 10x$$

Simplify:

$$h'(x) = -30x\cos^2(5x^2 - 1)\sin(5x^2 - 1)$$

5.1 Practice Problem

Work this out: Find the derivative of $f(x) = \sin^2(3x)$.

6 Using Leibniz Notation

Sometimes it's easier to use $\frac{dy}{dx}$ notation.

Example 8. *Find* $\frac{dy}{dx}$ *if* $y = (3 - 2x)^5$.

Solution:
Let $u = 3 - 2x$, *so* $y = u^5$
Then: $\frac{du}{dx} = -2$ *and* $\frac{dy}{du} = 5u^4$
By chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 5u^4 \cdot (-2) \\ &= -10(3 - 2x)^4\end{aligned}$$

Example 9. *Find* $\frac{dy}{dx}$ *if* $y = \tan(x^2 - 2x + 1)$.

Solution:
Let $u = x^2 - 2x + 1$, *so* $y = \tan u$
Then: $\frac{du}{dx} = 2x - 2$ *and* $\frac{dy}{du} = \sec^2 u$

$$\frac{dy}{dx} = \sec^2(x^2 - 2x + 1) \cdot (2x - 2)$$
$$= 2(x-1)\sec^2(x^2 - 2x + 1)$$

7 Applications

Example 10 (Velocity Problem)**.** *A particle's position is* $s(t) = \sin(2t^3)$*. Find its velocity at* $t = 1$*.*

Solution:
Velocity is $v(t) = s'(t)$*:*

$$v(t) = \cos(2t^3) \cdot \frac{d}{dt}(2t^3)$$
$$= \cos(2t^3) \cdot 6t^2$$
$$= 6t^2 \cos(2t^3)$$

At $t = 1$*:* $v(1) = 6(1)^2 \cos(2) = 6\cos(2)$

8 Summary

Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$
Strategy:

1. Identify outer and inner functions
2. Differentiate outer (leave inner alone)
3. Multiply by derivative of inner
4. For multiple compositions, work outside to inside

Common Patterns:

- $(g(x))^n$: $n(g(x))^{n-1} \cdot g'(x)$
- $\sin(g(x))$: $\cos(g(x)) \cdot g'(x)$
- $\cos(g(x))$: $-\sin(g(x)) \cdot g'(x)$
- $\tan(g(x))$: $\sec^2(g(x)) \cdot g'(x)$

Leibniz notation: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$