Implicit Differentiation
Section 3.8
Calculus I - Lecture Notes

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Motivating Question

How do we find $\frac{dy}{dx}$ when we can't solve for y?

Consider the circle $x^2 + y^2 = 25$. We could solve for $y = \pm\sqrt{25 - x^2}$, but that's messy. What if we could differentiate *without* solving for y first? That's **implicit differentiation**.

1 Explicit vs. Implicit Functions

Definition 1 (Explicit vs. Implicit)**.** ***Explicit function:*** *y is expressed entirely in terms of x.*

- *Example: $y = x^2 + 1$*

Implicit function: *The relationship between x and y is given by an equation, but y is not isolated.*

- *Example: $x^2 + y^2 = 25$ (defines y implicitly)*

Key insight: Many equations define y as a function of x even though we can't (or don't want to) solve for y explicitly.

Example: The circle $x^2 + y^2 = 25$ actually defines *two* functions:

$$y = \sqrt{25 - x^2} \quad \text{(top half)} \quad \text{and} \quad y = -\sqrt{25 - x^2} \quad \text{(bottom half)}$$

But we can work with the entire circle at once using implicit differentiation!

2 The Method of Implicit Differentiation

Strategy:

1. Differentiate both sides of the equation with respect to x
2. Remember: y is a function of x, so use the chain rule when differentiating terms with y
3. Collect all terms with $\frac{dy}{dx}$ on one side
4. Solve for $\frac{dy}{dx}$

Key rule: $\frac{d}{dx}(y^n) = ny^{n-1} \cdot \frac{dy}{dx}$ (chain rule!)

2.1 First Example

Example 1. *Find* $\frac{dy}{dx}$ *if* $x^2 + y^2 = 25$.

Solution:
Step 1: *Differentiate both sides with respect to* x*:*

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

Step 2: *Apply derivative rules:*

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

Note: $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$ *by the chain rule!*
Step 3: *Solve for* $\frac{dy}{dx}$*:*

$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Important: The answer $\frac{dy}{dx} = -\frac{x}{y}$ is in terms of *both* x and y. This is normal for implicit differentiation!

2.2 Practice Problem

Work this out: Find $\frac{dy}{dx}$ if $x^2 - y^2 = 4$.

3 More Complex Examples

Example 2. *Find* $\frac{dy}{dx}$ *if* $x^3 \sin y + y = 4x + 3$.

Solution:

Differentiate both sides:

$$\frac{d}{dx}(x^3 \sin y + y) = \frac{d}{dx}(4x + 3)$$

Left side (use product rule on $x^3 \sin y$*):*

$$\frac{d}{dx}(x^3) \cdot \sin y + x^3 \cdot \frac{d}{dx}(\sin y) + \frac{dy}{dx} = 4$$

For $\frac{d}{dx}(\sin y)$*, use chain rule:*

$$3x^2 \sin y + x^3 \cos y \cdot \frac{dy}{dx} + \frac{dy}{dx} = 4$$

Collect $\frac{dy}{dx}$ *terms on left:*

$$x^3 \cos y \cdot \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$$

Factor:

$$\frac{dy}{dx}(x^3 \cos y + 1) = 4 - 3x^2 \sin y$$

Solve:

$$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$$

Example 3. *Find* $\frac{dy}{dx}$ *if* $y^3 + x^3 - 3xy = 0$ *(folium of Descartes).*

Solution:

Differentiate both sides:

$$\frac{d}{dx}(y^3) + \frac{d}{dx}(x^3) - \frac{d}{dx}(3xy) = 0$$

Apply rules (product rule for 3xy):

$$3y^2\frac{dy}{dx} + 3x^2 - \left(3y + 3x\frac{dy}{dx}\right) = 0$$

Simplify:

$$3y^2\frac{dy}{dx} + 3x^2 - 3y - 3x\frac{dy}{dx} = 0$$

Collect $\frac{dy}{dx}$ *terms:*

$$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$

Factor:

$$\frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$$

Solve:

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}$$

3.1 Practice Problem

Work this out: Find $\frac{dy}{dx}$ if $6x^2 + 3y^2 = 12$.

Example 4. *Find* $\frac{dy}{dx}$ *if* $x^2y = y - 7$.

Solution:
Differentiate both sides:

$$\frac{d}{dx}(x^2y) = \frac{d}{dx}(y - 7)$$

Left side needs product rule:

$$\frac{d}{dx}(x^2) \cdot y + x^2 \cdot \frac{dy}{dx} = \frac{dy}{dx} - 0$$

$$2xy + x^2\frac{dy}{dx} = \frac{dy}{dx}$$

Collect $\frac{dy}{dx}$ *terms on left:*

$$x^2\frac{dy}{dx} - \frac{dy}{dx} = -2xy$$

Factor:

$$\frac{dy}{dx}(x^2 - 1) = -2xy$$

Solve:

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - 1}$$

Example 5. *Find* $\dfrac{dy}{dx}$ *if* $\tan(xy) = y$.

Solution:
Differentiate both sides:

$$\frac{d}{dx}[\tan(xy)] = \frac{dy}{dx}$$

Left side needs chain rule. Let $u = xy$*:*

$$\sec^2(xy) \cdot \frac{d}{dx}(xy) = \frac{dy}{dx}$$

For $\dfrac{d}{dx}(xy)$*, use product rule:*

$$\sec^2(xy) \cdot \left(y + x\frac{dy}{dx}\right) = \frac{dy}{dx}$$

Expand:

$$y\sec^2(xy) + x\sec^2(xy)\frac{dy}{dx} = \frac{dy}{dx}$$

Collect $\dfrac{dy}{dx}$ *terms:*

$$x\sec^2(xy)\frac{dy}{dx} - \frac{dy}{dx} = -y\sec^2(xy)$$

Factor:

$$\frac{dy}{dx}[x\sec^2(xy) - 1] = -y\sec^2(xy)$$

Solve:

$$\frac{dy}{dx} = \frac{-y\sec^2(xy)}{x\sec^2(xy) - 1}$$

Example 6. *Find* $\dfrac{dy}{dx}$ *if* $xy^2 + \sin(\pi y) - 2x^2 = 10$.

Solution:
Differentiate both sides:

$$\frac{d}{dx}(xy^2) + \frac{d}{dx}[\sin(\pi y)] - \frac{d}{dx}(2x^2) = 0$$

Product rule on xy^2*, chain rule on* $\sin(\pi y)$*:*

$$\left(y^2 + x \cdot 2y\frac{dy}{dx}\right) + \cos(\pi y) \cdot \pi\frac{dy}{dx} - 4x = 0$$

Simplify:

$$y^2 + 2xy\frac{dy}{dx} + \pi\cos(\pi y)\frac{dy}{dx} - 4x = 0$$

Collect $\frac{dy}{dx}$ *terms:*

$$2xy\frac{dy}{dx} + \pi\cos(\pi y)\frac{dy}{dx} = 4x - y^2$$

Factor:

$$\frac{dy}{dx}[2xy + \pi\cos(\pi y)] = 4x - y^2$$

Solve:

$$\frac{dy}{dx} = \frac{4x - y^2}{2xy + \pi\cos(\pi y)}$$

4 Finding Tangent Lines

Implicit differentiation is especially useful for finding tangent lines to curves.

Example 7. *Find the equation of the tangent line to* $x^2 + y^2 = 25$ *at* $(3, -4)$*.*

Solution:
From our first example, we know $\frac{dy}{dx} = -\frac{x}{y}$.
At $(3, -4)$*:*

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$$

Using point-slope form with point $(3, -4)$ *and slope* $\frac{3}{4}$*:*

$$y - (-4) = \frac{3}{4}(x - 3)$$

$$y + 4 = \frac{3}{4}x - \frac{9}{4}$$

$$y = \frac{3}{4}x - \frac{25}{4}$$

Example 8. *Find the tangent line to* $y^3 + x^3 - 3xy = 0$ *at* $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Solution:
From earlier, $\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$.
At $\left(\frac{3}{2}, \frac{3}{2}\right)$*:*

$$\frac{dy}{dx} = \frac{\frac{3}{2} - \left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - \frac{3}{2}} = \frac{\frac{3}{2} - \frac{9}{4}}{\frac{9}{4} - \frac{3}{2}} = \frac{-\frac{3}{4}}{\frac{3}{4}} = -1$$

Equation: $y - \frac{3}{2} = -1\left(x - \frac{3}{2}\right)$ *or* $y = -x + 3$

4.1 Practice Problem

Work this out: Find the tangent line to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

5 Second Derivatives

We can find second derivatives implicitly too!

Example 9. *Find* $\dfrac{d^2y}{dx^2}$ *if* $x^2 + y^2 = 25$.

Solution:
We already know $\dfrac{dy}{dx} = -\dfrac{x}{y}$.
Differentiate both sides again (use quotient rule):

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{x}{y}\right) \\ &= -\frac{(1)(y) - (x)\frac{dy}{dx}}{y^2} \\ &= -\frac{y - x\frac{dy}{dx}}{y^2}\end{aligned}$$

Substitute $\dfrac{dy}{dx} = -\dfrac{x}{y}$*:*

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} \\ &= -\frac{y + \frac{x^2}{y}}{y^2} \\ &= -\frac{y^2 + x^2}{y^3}\end{aligned}$$

Since $x^2 + y^2 = 25$*:*

$$\frac{d^2y}{dx^2} = -\frac{25}{y^3}$$

6 Applications

Example 10 (Video Game Problem)**.** *A rocket travels in an elliptical orbit:* $4x^2 + 25y^2 = 100$. *It fires a missile along a tangent line when at* $\left(3, \frac{8}{5}\right)$. *Where does the missile hit the* x*-axis?*

Solution:
Find $\dfrac{dy}{dx}$ *by differentiating:*

$$8x + 50y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{8x}{50y} = -\frac{4x}{25y}$$

At $\left(3, \frac{8}{5}\right)$*:*

$$\frac{dy}{dx} = -\frac{4(3)}{25 \cdot \frac{8}{5}} = -\frac{12}{40} = -\frac{3}{10}$$

Tangent line: $y - \frac{8}{5} = -\frac{3}{10}(x - 3)$
Simplify: $y = -\frac{3}{10}x + \frac{9}{10} + \frac{8}{5} = -\frac{3}{10}x + \frac{25}{10}$
When $y = 0$*:* $0 = -\frac{3}{10}x + \frac{5}{2}$*, so* $x = \frac{25}{3}$
The missile hits at $\left(\frac{25}{3}, 0\right)$*.*

7 Summary

Implicit Differentiation Steps:

1. Differentiate both sides with respect to x
2. Use chain rule: $\dfrac{d}{dx}(y^n) = ny^{n-1}\dfrac{dy}{dx}$
3. Use product rule when needed: $\dfrac{d}{dx}(xy) = y + x\dfrac{dy}{dx}$
4. Collect all $\dfrac{dy}{dx}$ terms on one side
5. Factor out $\dfrac{dy}{dx}$
6. Solve for $\dfrac{dy}{dx}$

Key points:

- The answer will usually involve both x and y
- Treat y as a function of x (use chain rule!)
- Can be used to find tangent lines to curves that aren't functions
- Can find second derivatives by differentiating $\dfrac{dy}{dx}$ again

Common terms:

- $\dfrac{d}{dx}(y) = \dfrac{dy}{dx}$

- $\dfrac{d}{dx}(y^2) = 2y\dfrac{dy}{dx}$
- $\dfrac{d}{dx}(y^3) = 3y^2\dfrac{dy}{dx}$
- $\dfrac{d}{dx}(xy) = y + x\dfrac{dy}{dx}$
- $\dfrac{d}{dx}(\sin y) = \cos y \cdot \dfrac{dy}{dx}$