Related Rates & Linear Approximations
Sections 4.1–4.2
Calculus I - Lecture Notes

February 24, 2026

Review of Chapter 3

Two problems to warm up. Work them out before looking at the solutions.

Example 1 (Quotient Rule + Trig)**.** *Find the derivative of* $f(x) = \dfrac{\sin x}{x^2+1}$.

Solution:
Use the quotient rule $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$ *with* $u = \sin x$ *and* $v = x^2 + 1$*:*

$$u' = \cos x, \qquad v' = 2x$$

$$f'(x) = \frac{\cos x \cdot (x^2+1) - \sin x \cdot 2x}{(x^2+1)^2} = \frac{(x^2+1)\cos x - 2x \sin x}{(x^2+1)^2}$$

Example 2 (Chain Rule: Particle Motion)**.** *A particle moves along a line with position* $s(t) = (t^2-1)^3$*. At* $t = 2$*:*

- *Is the particle moving right or left?*
- *Is it speeding up or slowing down?*

Solution:
Velocity $v(t) = s'(t)$*. Apply the chain rule:*

$$v(t) = 3(t^2-1)^2 \cdot 2t = 6t(t^2-1)^2$$

At $t = 2$*:* $\quad t^2 - 1 = 3$

$$v(2) = 6(2)(3)^2 = 108 > 0 \implies \textit{\textbf{moving right}}$$

Acceleration $a(t) = v'(t)$. *Use the product rule on* $v(t) = 6t(t^2-1)^2$*:*

$$a(t) = 6(t^2-1)^2 + 6t \cdot 2(t^2-1)(2t) = 6(t^2-1)^2 + 24t^2(t^2-1)$$

At $t = 2$*:*

$$a(2) = 6(3)^2 + 24(4)(3) = 54 + 288 = 342 > 0$$

Since $v(2) > 0$ *and* $a(2) > 0$ *have the* ***same sign*** $\implies$ ***speeding up****.*

Motivating Question

If two quantities are linked by an equation, how fast does one change when we know how fast the other changes?

When a balloon is inflated, both its radius and volume grow together. Knowing one rate of change lets us find the other — this is the power of **related rates** and **linear approximation**.

1 Related Rates (Section 4.1)

1.1 The Core Idea

If two quantities x and y are related by an equation, we can differentiate both sides with respect to time t to relate their rates $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Theorem 1 (Related Rates Strategy)**.**

1. *Assign variables to all changing quantities. Draw a figure.*
2. *Write an equation relating those variables.*
3. *Differentiate both sides with respect to* t *(using the chain rule).*
4. *Substitute the known values and solve for the unknown rate.*

Warning: *Do not substitute known values before differentiating — a quantity that is "fixed at a moment" still has a nonzero derivative!*

1.2 Example: Inflating a Balloon

Example 3. *A spherical balloon is filled with air at 2* cm^3/sec*. How fast is the radius increasing when* $r = 3$ *cm?*

Solution:

The volume of a sphere is $V = \frac{4}{3}\pi r^3$*. Both* V *and* r *depend on* t*, so differentiate:*

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

We know $\frac{dV}{dt} = 2$ *and* $r = 3$*:*

$$2 = 4\pi(3)^2 \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{1}{18\pi} \text{ cm/sec}$$

1.3 Example: Airplane and Observer

Example 4. *An airplane flies at 4000 ft altitude at 600 ft/sec horizontally away from a person on the ground. The person is directly below the plane's starting point. How fast is the distance* s *between them increasing when the horizontal distance is* $x = 3000$ *ft?*

Solution:
By the Pythagorean theorem: $x^2 + 4000^2 = s^2$*.*
Differentiate with respect to t*:*

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt} \implies x\frac{dx}{dt} = s\frac{ds}{dt}$$

When $x = 3000$*:* $s = \sqrt{3000^2 + 4000^2} = 5000$ *ft. With* $\frac{dx}{dt} = 600$*:*

$$(3000)(600) = (5000)\frac{ds}{dt} \implies \frac{ds}{dt} = 360 \text{ ft/sec}$$

1.4 Practice Problem

Work this out: A 10-ft ladder leans against a wall. The top slides down at 2 ft/sec. How fast is the bottom moving away from the wall when the bottom is 5 ft from the wall?

Solution:

Let x = distance of the bottom from the wall and y = height of the top on the wall. The ladder length is constant, so by the Pythagorean theorem:

$$x^2 + y^2 = 100$$

Differentiate with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

We know $x = 5$ and $\frac{dy}{dt} = -2$ (negative because the top is sliding *down*). First find y:

$$y = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$$

Substitute:

$$2(5)\frac{dx}{dt} + 2(5\sqrt{3})(-2) = 0 \implies 10\frac{dx}{dt} = 20\sqrt{3} \implies \frac{dx}{dt} = 2\sqrt{3} \approx 3.46 \text{ ft/sec}$$

1.5 Example: Rocket Launch

Example 5. *A rocket rises vertically. A camera 5000 ft away tracks it. When the rocket is 1000 ft high with velocity 600 ft/sec, find* $\frac{d\theta}{dt}$.

Solution:
Let h = *height and* θ = *camera angle. Then* $\tan\theta = \frac{h}{5000}$, *so* $h = 5000\tan\theta$.
Differentiate:

$$\frac{dh}{dt} = 5000\sec^2\theta\,\frac{d\theta}{dt}$$

When $h = 1000$*: the hypotenuse is* $c = \sqrt{1000^2 + 5000^2} = 1000\sqrt{26}$, *so*

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}$$

Substituting $\frac{dh}{dt} = 600$*:*

$$600 = 5000 \cdot \frac{26}{25} \cdot \frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{3}{26} \text{ rad/sec}$$

1.6 Example: Water Draining from a Cone

Example 6. *Water drains from a cone-shaped funnel at* 0.03 *ft*^{3}*/sec. The funnel has height 2 ft and top radius 1 ft. How fast is the water level falling when the water is* $\frac{1}{2}$ *ft deep?*

Solution:
Let h = *height of water and* r = *radius of the water surface. The volume of a cone is:*

$$V = \frac{1}{3}\pi r^2 h$$

Key step — eliminate r***:*** *The funnel's full dimensions give ratio* $\frac{r}{h} = \frac{1}{2}$, *so* $r = \frac{h}{2}$.
Substitute:

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now differentiate with respect to t*:*

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

Water is leaving, so $\frac{dV}{dt} = -0.03$. *At* $h = \frac{1}{2}$*:*

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt} = \frac{\pi}{16}\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{-0.48}{\pi} \approx -0.153 \text{ ft/sec}$$

The water level is falling at about 0.153 *ft/sec.*

2 Linear Approximations and Differentials (Section 4.2)

2.1 Linearization

Near a point $x = a$, a differentiable function looks like its tangent line. We can use this line to approximate function values.

Definition 1 (Linear Approximation / Linearization)**.** *The **linearization** of f at $x = a$ is*

$$L(x) = f(a) + f'(a)(x - a)$$

For x near a: $f(x) \approx L(x)$.

Example 7. *Find the linearization of $f(x) = \sqrt{x}$ at $x = 9$ and estimate $\sqrt{9.1}$.*

Solution:
$f(9) = 3$ *and* $f'(x) = \dfrac{1}{2\sqrt{x}} \Rightarrow f'(9) = \dfrac{1}{6}$

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

$$\sqrt{9.1} \approx L(9.1) = 3 + \frac{1}{6}(0.1) \approx 3.0167$$

(Calculator gives 3.0166 *— very close!)*

Example 8. *Find the linearization of $f(x) = \sin x$ at $x = \pi/3$ and estimate* $\sin(62°)$.

Solution:
$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$$L(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

Convert $62° = \dfrac{62\pi}{180}$ *rad, so* $x - \dfrac{\pi}{3} = \dfrac{2\pi}{180}$*:*

$$\sin(62°) \approx \frac{\sqrt{3}}{2} + \frac{\pi}{180} \approx 0.8835$$

2.2 Practice Problem

Work this out: Find the linearization of $f(x) = (1+x)^n$ at $x = 0$ and use it to approximate $(1.01)^3$.

Solution:
$f(0) = (1 + 0)^n = 1$ and $f'(x) = n(1 + x)^{n-1} \Rightarrow f'(0) = n$

$$L(x) = 1 + nx$$

For $(1.01)^3$, set $n = 3$ and $x = 0.01$:

$$(1.01)^3 \approx L(0.01) = 1 + 3(0.01) = 1.03$$

(The exact value is 1.030301 — very close!)

2.3 Differentials

Definition 2 (Differential)**.** *For $y = f(x)$, the **differential** dy is defined by*

$$dy = f'(x)\,dx$$

where dx is any nonzero real number. This approximates the actual change $\Delta y = f(x + dx) - f(x)$:

$$\Delta y \approx dy \quad \text{when } dx \text{ is small}$$

Example 9. *Let $y = x^2 + 2x$. Compute Δy and dy at $x = 3$, $dx = 0.1$.*

Solution:
Actual change: $\Delta y = f(3.1) - f(3) = [(3.1)^2 + 2(3.1)] - [9 + 6] = 0.81$
Differential: $f'(x) = 2x + 2$, *so*

$$dy = f'(3)(0.1) = (8)(0.1) = 0.8$$

The approximation $dy = 0.8$ is close to $\Delta y = 0.81$.

Practice: For $y = x^2 + 2x$, find Δy and dy at $x = 3$, $dx = 0.2$.

Solution:
Actual change:

$$\Delta y = f(3.2) - f(3) = [(3.2)^2 + 2(3.2)] - [9 + 6] = [10.24 + 6.4] - 15 = 1.64$$

Differential: $f'(x) = 2x + 2$, so

$$dy = f'(3)(0.2) = (8)(0.2) = 1.6$$

The differential $dy = 1.6$ approximates the actual change $\Delta y = 1.64$ with a small error of 0.04.

2.4 Error Estimation with Differentials

If a measurement has absolute error dx, the **propagated error** in a computed quantity $f(x)$ is approximated by:

$$\Delta y \approx dy = f'(x)\,dx$$

The **relative error** is $\frac{dy}{y}$ and the **percentage error** is $\frac{dy}{y} \times 100\%$.

Example 10. *A cube's side is measured as 5 cm with accuracy ± 0.1 cm. Estimate the error in volume.*

Solution:
$V = x^3 \Rightarrow dV = 3x^2\,dx$. *With $x = 5$ and $|dx| \leq 0.1$:*

$$|dV| \leq 3(5)^2(0.1) = 7.5 \ cm^3$$

(Actual error range: 117.649 to 132.651 vs. computed $V = 125$, giving $|\Delta V| \leq 7.651 \ cm^3$.)

Example 11 (Relative and Percentage Error). *A sphere is measured to have radius $r = 6$ cm with a possible error of ± 0.2 cm. Estimate the relative error and percentage error in the computed volume.*

Solution:
The volume of a sphere is $V = \frac{4}{3}\pi r^3$, so:

$$dV = 4\pi r^2\, dr$$

With $r = 6$ and $|dr| \leq 0.2$:

$$|dV| \leq 4\pi(6)^2(0.2) = 4\pi(36)(0.2) = 28.8\pi \ cm^3$$

*To find the **relative error**, divide dV by V:*

$$\frac{dV}{V} = \frac{4\pi r^2\, dr}{\frac{4}{3}\pi r^3} = \frac{3\, dr}{r}$$

At $r = 6$ with $|dr| \leq 0.2$:

$$\left|\frac{dV}{V}\right| \leq \frac{3(0.2)}{6} = 0.1$$

*The **relative error** is 0.1 and the **percentage error** is **10%**.*

Note: *The nice cancellation $\frac{dV}{V} = \frac{3\, dr}{r}$ shows that a 1% error in the radius produces a 3% error in volume — always true for spheres!*

3 Summary

Related Rates (4.1):

1. Relate variables with an equation
2. Differentiate with respect to t
3. Substitute known values *after* differentiating

Linear Approximation (4.2):

$$f(x) \approx L(x) = f(a) + f'(a)(x - a) \quad \text{for } x \text{ near } a$$

Differentials:

$$dy = f'(x)\, dx \approx \Delta y \quad \text{(propagated error estimate)}$$

Key connection: Both topics use the derivative as a local rate — related rates use it across time, linear approximation uses it across space.