Maxima, Minima & the Mean Value Theorem
Sections 4.3–4.4
Calculus I - Lecture Notes

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Motivating Question

Where does a function reach its highest and lowest values, and how can we guarantee that such points exist?

Finding maximum and minimum values is one of the most powerful applications of calculus — from maximizing profit to minimizing materials used in manufacturing.

1 Maxima and Minima (Section 4.3)

1.1 Absolute vs. Local Extrema

Definition 1 (Absolute Extrema)**.** *Let f be defined on an interval I and let $c \in I$.*

- *f has an **absolute maximum** on I at c if $f(c) \geq f(x)$ for all $x \in I$.*
- *f has an **absolute minimum** on I at c if $f(c) \leq f(x)$ for all $x \in I$.*

Definition 2 (Local Extrema)**.** *f has a **local maximum** at c if $f(c) \geq f(x)$ for all x* near *c. Similarly for a **local minimum**. Think of these as "hilltops" and "valleys" that may not be the global highest or lowest point.*

Theorem 1 (Extreme Value Theorem)**.** *If f is continuous on a closed, bounded interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.*

Key conditions: The interval must be *closed* (includes endpoints) and f must be *continuous*. If either condition fails, the theorem may not apply.

1.2 Critical Points and Fermat's Theorem

Definition 3 (Critical Point)**.** *Let c be an interior point of the domain of f. We call c a* ***critical point*** *of f if:*

$$f'(c) = 0 \quad \textit{or} \quad f'(c) \textit{ is undefined}$$

Theorem 2 (Fermat's Theorem)**.** *If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.*

Important warning: The converse is *not* true. A critical point is only a *candidate* for a local extremum — it does not guarantee one. For example, $f(x) = x^3$ has $f'(0) = 0$ but no local extremum at $x = 0$.

1.3 Example: Finding Critical Points

Example 1. *Find all critical points of $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$.*

Solution:
$f'(x) = x^2 - 5x + 4$. *This is defined for all x, so we only solve $f'(x) = 0$:*

$$x^2 - 5x + 4 = (x-4)(x-1) = 0 \implies x = 1 \textit{ and } x = 4$$

Both are critical points. Evaluating: $f(1) = \frac{1}{3} - \frac{5}{2} + 4 = \frac{11}{6}$ (local max) and $f(4) = \frac{64}{3} - 40 + 16 = -\frac{8}{3}$ (local min).

1.4 Locating Absolute Extrema on a Closed Interval

Theorem 3 (Location of Absolute Extrema)**.** *Let f be continuous on $[a, b]$. The absolute maximum and minimum of f must occur either at* ***endpoints*** *of $[a, b]$ or at* ***critical points*** *inside (a, b).*

Strategy — the Closed Interval Method:

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points in (a, b) and evaluate f there.
3. The largest value from steps 1 and 2 is the absolute maximum; the smallest is the absolute minimum.

1.5 Example: Closed Interval Method

Example 2. *Find the absolute maximum and minimum of $f(x) = -x^2 + 3x - 2$ on $[1, 3]$.*

Solution:
Step 1 — Endpoints:

$$f(1) = -1 + 3 - 2 = 0 \qquad f(3) = -9 + 9 - 2 = -2$$

Step 2 — Critical points: *$f'(x) = -2x + 3 = 0 \implies x = \frac{3}{2}$, which is in $[1, 3]$.*

$$f\left(\tfrac{3}{2}\right) = -\tfrac{9}{4} + \tfrac{9}{2} - 2 = \tfrac{1}{4}$$

Step 3 — Compare:

x	$f(x)$	
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	***Absolute maximum***
3	-2	***Absolute minimum***

1.6 Example: Critical Point Where Derivative is Undefined

Example 3. *Find all critical points of $f(x) = x^{2/3}$ and classify them.*

Solution:
$f'(x) = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}}$

*The derivative is never zero, but it is **undefined at** $x = 0$. Since $x = 0$ is in the domain of f, it is a critical point.*

*Checking values near $x = 0$: for $x < 0$, $f'(x) < 0$ (decreasing); for $x > 0$, $f'(x) > 0$ (increasing). The function decreases then increases, so $x = 0$ is a **local minimum** with $f(0) = 0$.*

Key lesson: *Always check where f' is undefined — these are critical points too, and they can produce sharp corners (cusps) on the graph rather than smooth peaks or valleys.*

1.7 Practice Problem

Work this out: Find the absolute maximum and minimum of $f(x) = x^2 - 4x + 3$ on $[1, 4]$.

Solution:
Step 1 — Endpoints:

$$f(1) = 1 - 4 + 3 = 0 \qquad f(4) = 16 - 16 + 3 = 3$$

Step 2 — Critical points: $f'(x) = 2x - 4 = 0 \implies x = 2$, which is in $[1, 4]$.

$$f(2) = 4 - 8 + 3 = -1$$

Step 3 — Compare:

x	$f(x)$	
1	0	
2	-1	**Absolute minimum**
4	3	**Absolute maximum**

2 The Mean Value Theorem (Section 4.4)

2.1 Rolle's Theorem

Rolle's Theorem is the special case that motivates the Mean Value Theorem.

Theorem 4 (Rolle's Theorem)**.** *Let f be continuous on $[a, b]$, differentiable on (a, b), and suppose $f(a) = f(b)$. Then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.*

Intuition: If a function starts and ends at the same height, it must have a horizontal tangent somewhere in between — either a peak or a valley.

Proof: Let $k = f(a) = f(b)$. We consider three cases.

Case 1: $f(x) = k$ for all $x \in (a, b)$. Then f is constant, so $f'(x) = 0$ everywhere on (a, b) and we are done.

Case 2: $f(x) > k$ for some $x \in (a, b)$. Since f is continuous on the closed interval $[a, b]$, the Extreme Value Theorem guarantees an absolute maximum at some point c. Since $f(x) > k = f(a) = f(b)$ somewhere inside, the maximum cannot occur at either endpoint — it must occur at an interior point $c \in (a, b)$. Since f has a local maximum at c and is differentiable there, Fermat's Theorem gives $f'(c) = 0$.

Case 3: $f(x) < k$ for some $x \in (a, b)$. By the same argument with minimum replacing maximum, there exists an interior c where $f'(c) = 0$. □

From Rolle's Theorem to the MVT. The MVT handles the general case where $f(a) \neq f(b)$. The trick is to subtract the secant line from f, creating a new function g that *does* satisfy Rolle's conditions.

Define:

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]$$

The bracketed term is just the equation of the secant line through $(a, f(a))$ and $(b, f(b))$. So $g(x)$ measures the vertical gap between f and that secant line. Check the endpoints:

$$g(a) = f(a) - f(a) = 0 \qquad g(b) = f(b) - f(b) = 0$$

Since $g(a) = g(b) = 0$ and g inherits continuity and differentiability from f, Rolle's Theorem applies to g. Therefore there exists $c \in (a, b)$ with $g'(c) = 0$.

Differentiating g:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Setting $g'(c) = 0$:

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \implies f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \square$$

This is exactly the MVT conclusion. Geometrically, subtracting the secant line "flattens" the picture so that Rolle's horizontal tangent becomes a tangent parallel to the secant.

Example 4. *Verify Rolle's Theorem for $f(x) = x^2 + 2x$ on $[-2, 0]$ and find c.*

Solution:

f is a polynomial, so it is continuous and differentiable everywhere. Check: $f(-2) = 4 - 4 = 0$ and $f(0) = 0$. Since $f(-2) = f(0)$, Rolle's Theorem applies.

$f'(x) = 2x + 2 = 0 \implies x = -1$

Since $-1 \in (-2, 0)$, we have $c = -1$. ✓

2.2 The Mean Value Theorem

The Mean Value Theorem generalizes Rolle's Theorem by removing the requirement that $f(a) = f(b)$.

Theorem 5 (Mean Value Theorem)**.** *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometric meaning: At some interior point c, the slope of the *tangent line* equals the slope of the *secant line* connecting the endpoints $(a, f(a))$ and $(b, f(b))$.

Real-world meaning: If you drive 90 miles in 2 hours, your average speed is 45 mph. The MVT guarantees there was at least one moment when your speedometer read *exactly* 45 mph.

2.3 Example: Applying the Mean Value Theorem

Example 5. *For $f(x) = \sqrt{x}$ on $[0, 9]$, find all values c guaranteed by the Mean Value Theorem.*

Solution:

f is continuous on $[0, 9]$ and differentiable on $(0, 9)$, so MVT applies.

Slope of secant line:

$$\frac{f(9) - f(0)}{9 - 0} = \frac{3 - 0}{9} = \frac{1}{3}$$

Set $f'(c) = \frac{1}{3}$. Since $f'(x) = \frac{1}{2\sqrt{x}}$:

$$\frac{1}{2\sqrt{c}} = \frac{1}{3} \implies 2\sqrt{c} = 3 \implies \sqrt{c} = \frac{3}{2} \implies c = \frac{9}{4}$$

Since $\frac{9}{4} \in (0, 9)$, the MVT is satisfied at $c = \frac{9}{4}$.

2.4 Example: MVT Applied to Velocity

Example 6. *A rock is dropped from 100 ft. Its position is $s(t) = -16t^2 + 100$ ft. Find the time when the rock's instantaneous velocity equals its average velocity over the entire fall.*

Solution:
Step 1 — Find when the rock hits the ground:

$$-16t^2 + 100 = 0 \implies t^2 = \frac{100}{16} \implies t = \frac{5}{2} \text{ sec}$$

Step 2 — Find average velocity *over $\left[0, \frac{5}{2}\right]$:*

$$v_{avg} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{0 - 100}{5/2} = -40 \text{ ft/sec}$$

Step 3 — Apply MVT. *Since $s(t)$ is continuous on $[0, 5/2]$ and differentiable on $(0, 5/2)$, the MVT guarantees a c where $s'(c) = -40$.*
$s'(t) = -32t$, so:

$$-32c = -40 \implies c = \frac{5}{4} \text{ sec}$$

At $t = \frac{5}{4}$ seconds, the instantaneous velocity equals the average velocity of -40 ft/sec.

2.5 Practice Problem

Work this out: For $f(x) = x^2$ on $[1, 3]$, find all values c guaranteed by the Mean Value Theorem.

Solution:
Slope of secant line:

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4$$

Set $f'(c) = 4$. Since $f'(x) = 2x$:

$$2c = 4 \implies c = 2$$

Since $2 \in (1, 3)$, the MVT is satisfied at $c = 2$. ✓

2.6 Three Important Corollaries

The MVT has powerful consequences used throughout the rest of the course.
Corollary 1: If $f'(x) = 0$ for all x on an interval I, then f is constant on I.
Corollary 2: If $f'(x) = g'(x)$ for all x on I, then $f(x) = g(x) + C$ for some constant C.
Corollary 3 (Increasing/Decreasing Test):

- If $f'(x) > 0$ for all $x \in (a, b)$, then f is **increasing** on $[a, b]$.
- If $f'(x) < 0$ for all $x \in (a, b)$, then f is **decreasing** on $[a, b]$.

3 Summary

Maxima and Minima (4.3):

1. A **critical point** satisfies $f'(c) = 0$ or $f'(c)$ undefined
2. Local extrema can only occur at critical points (Fermat's Theorem)
3. On a closed interval $[a, b]$: use the **Closed Interval Method** — evaluate f at endpoints and all critical points, then compare

Mean Value Theorem (4.4):

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{for some } c \in (a, b)$$

1. **Rolle's Theorem** is the special case where $f(a) = f(b)$, giving $f'(c) = 0$
2. The MVT connects average rate of change to instantaneous rate of change
3. Key consequence: the sign of f' tells us where f is increasing or decreasing