Introduction to Derivatives
Sections 3.1 & 3.2
Calculus I - Lecture Notes

February 3, 2026

Motivating Question

How do we measure instantaneous rate of change?

We've been studying limits and using them to find slopes of tangent lines. Today we formalize this into the most important concept in calculus: **the derivative**.

1 The Derivative: Definition and Examples

1.1 The Derivative at a Point

Recall from our work with limits that we found slopes of tangent lines by taking limits of difference quotients. This process is so important we give it a special name.

Definition 1 (Derivative at a Point)**.** *The* ***derivative of*** $f(x)$ ***at*** a*, denoted* $f'(a)$*, is:*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

Equivalently: $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

What does $f'(a)$ tell us?

- The slope of the tangent line to f at $x = a$
- The instantaneous rate of change of f at $x = a$

Example 1 (Computing a Derivative)**.** *For* $f(x) = x^2$*, find* $f'(3)$*.*

Solution:

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{(3+h)^2-9}{h} \\ &= \lim_{h\to 0} \frac{9+6h+h^2-9}{h} \\ &= \lim_{h\to 0} \frac{6h+h^2}{h} \\ &= \lim_{h\to 0}(6+h) = 6 \end{aligned}$$

So $f'(3) = 6$. This means the slope of the tangent line at $x = 3$ is 6.

Example 2 (Another Derivative Calculation)**.** *For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$.*

Solution:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{[3(2+h)^2-4(2+h)+1]-5}{h} \\ &= \lim_{h\to 0} \frac{3(4+4h+h^2)-8-4h+1-5}{h} \\ &= \lim_{h\to 0} \frac{12h+3h^2-4h}{h} \\ &= \lim_{h\to 0}(8+3h) = 8 \end{aligned}$$

1.2 Practice Problem

Work this out: For $f(x) = x^2 + 3x + 2$, find $f'(1)$.

1.3 Connection to Tangent Lines

Now we can connect this to what we learned about tangent lines earlier.

To find the equation of a tangent line at $x = a$:

1. Compute $f'(a)$ to get the slope
2. Use the point $(a, f(a))$
3. Write the equation: $y - f(a) = f'(a)(x - a)$

Example 3 (Finding a Tangent Line)**.** *Find the equation of the line tangent to* $f(x) = x^2$ *at* $x = 3$*.*

Solution:
From our earlier example, we know $f'(3) = 6$*.*
The tangent line passes through $(3, f(3)) = (3, 9)$ *with slope 6.*
Equation: $y - 9 = 6(x - 3)$*, which simplifies to* $y = 6x - 9$*.*

1.4 Velocity and Instantaneous Rate of Change

The derivative has important physical interpretations.

If $s(t)$ is the position of an object at time t:

- **Average velocity** over $[a, t]$: $v_{\text{ave}} = \dfrac{s(t) - s(a)}{t - a}$
- **Instantaneous velocity** at $t = a$: $v(a) = s'(a)$

Example 4 (Instantaneous Velocity)**.** *A rock is dropped from 64 feet. Its height is* $s(t) = -16t^2 + 64$*. Find velocity at* $t = 1$*.*

Solution:

$$\begin{aligned}
v(1) = s'(1) &= \lim_{h\to 0} \frac{s(1+h) - s(1)}{h} \\
&= \lim_{h\to 0} \frac{[-16(1+h)^2 + 64] - 48}{h} \\
&= \lim_{h\to 0} \frac{-16 - 32h - 16h^2 + 64 - 48}{h} \\
&= \lim_{h\to 0} \frac{-32h - 16h^2}{h} \\
&= \lim_{h\to 0} (-32 - 16h) = -32 \text{ ft/s}
\end{aligned}$$

The rock is falling at 32 ft/s (negative indicates downward).

2 The Derivative as a Function

2.1 The Derivative Function

Definition 2 (Derivative Function)**.** *The* ***derivative function*** f' *is defined by:*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

A function is ***differentiable at*** a *if* $f'(a)$ *exists.*

Example 5 (Derivative of $\sqrt{x}$)**.** *Find $f'(x)$ for $f(x) = \sqrt{x}$.*

Solution:

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$
$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

Example 6 (Derivative of a Quadratic)**.** *Find $f'(x)$ for $f(x) = x^2 - 2x$.*

Solution:

$$f'(x) = \lim_{h \to 0} \frac{[(x+h)^2 - 2(x+h)] - (x^2 - 2x)}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h \to 0}(2x + h - 2) = 2x - 2$$

2.2 Practice Problem

Work this out: Find $f'(x)$ for $f(x) = x^2$.

2.3 Notation for Derivatives

For $y = f(x)$, the derivative can be written as:

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}(f(x))$$

The notation $\frac{dy}{dx}$ (Leibniz notation) reminds us that the derivative is the limit of $\frac{\Delta y}{\Delta x}$.

2.4 Graphing Derivatives

Key relationships:

- Where f is **increasing**, $f' > 0$
- Where f is **decreasing**, $f' < 0$
- Where f has a **horizontal tangent**, $f' = 0$

2.5 Differentiability and Continuity

Theorem 1 (Differentiability Implies Continuity)**.** *If f is differentiable at a, then f is continuous at a.*

Important: The converse is false! Continuity does not guarantee differentiability.

2.6 When Is a Function Not Differentiable?

A function fails to be differentiable at a if:

1. **Discontinuity:** f is not continuous at a
2. **Corner/Cusp:** Example: $f(x) = |x|$ at $x = 0$

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \neq 1 = \lim_{x \to 0^+} \frac{|x|}{x}$$

3. **Vertical tangent:** Example: $f(x) = \sqrt[3]{x}$ at $x = 0$

$$f'(0) = \lim_{x \to 0} \frac{1}{x^{2/3}} = +\infty$$

2.7 Practice Problem

Work this out: Find values of a and b that make

$$f(x) = \begin{cases} ax + b & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$$

continuous and differentiable at $x = 3$.

Hint: For continuity, set $\lim_{x \to 3^-} f(x) = f(3)$*. For differentiability, set the left and right derivatives equal.*

2.8 Higher-Order Derivatives

The derivative of a derivative is called the **second derivative**.

Notation: $f''(x)$, y'', $\dfrac{d^2y}{dx^2}$

Example 7 (Second Derivative)**.** *For* $f(x) = 2x^2 - 3x + 1$*, find* $f''(x)$*.*

Solution:
First: $f'(x) = 4x - 3$
Then: $f''(x) = 4$

Physical Interpretation:

- Position: $s(t)$
- Velocity: $v(t) = s'(t)$
- Acceleration: $a(t) = v'(t) = s''(t)$

Example 8 (Acceleration)**.** *For* $s(t) = 3t^2 - 4t + 1$ *meters, find the acceleration.*

Solution:
$v(t) = s'(t) = 6t - 4 \ m/s$
$a(t) = s''(t) = 6 \ m/s^2$

3 Summary

The Derivative:

- At a point: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ gives slope and instantaneous rate of change
- As a function: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ gives a formula for all derivatives
- Differentiability implies continuity (but not vice versa)
- Functions fail to be differentiable at: discontinuities, corners, vertical tangents
- Second derivative $f''(x)$ measures how the rate of change is changing (e.g., acceleration)