

Differentiation Rules

Section 3.3

Calculus I - Lecture Notes

February 5, 2026

Motivating Question

Do we really need to use limits every time we find a derivative?

Using the limit definition for every derivative would be tedious! Today we develop **short-cut rules** that make finding derivatives much faster and easier.

1 The Basic Rules

1.1 The Constant Rule

What's the derivative of a constant function?

Theorem 1 (Constant Rule). *If $f(x) = c$ where c is a constant, then*

$$f'(x) = 0 \quad \text{or} \quad \frac{d}{dx}(c) = 0$$

Why? A constant function is a horizontal line, which has slope 0.

Proof:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0$$

Example 1. Find $\frac{d}{dx}(8)$.

Solution: $\frac{d}{dx}(8) = 0$

1.2 The Power Rule

This is the most important differentiation rule!

Theorem 2 (Power Rule). *If $f(x) = x^n$ where n is a positive integer, then*

$$f'(x) = nx^{n-1} \quad \text{or} \quad \frac{d}{dx}(x^n) = nx^{n-1}$$

Pattern: Bring down the exponent as a coefficient, then decrease the exponent by 1.

Example 2. Find $\frac{d}{dx}(x^3)$.

Solution:

$$\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$$

Example 3. Find the derivative of $f(x) = x^{10}$.

Solution:

$$f'(x) = 10x^9$$

1.3 Practice Problem

Work this out: Find $\frac{d}{dx}(x^7)$.

2 Combining Basic Rules

2.1 Sum, Difference, and Constant Multiple Rules

Theorem 3 (Sum, Difference, and Constant Multiple Rules). *Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant.*

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Constant Multiple Rule: $\frac{d}{dx}[kf(x)] = k \frac{d}{dx}[f(x)]$

In words:

- The derivative of a sum is the sum of the derivatives
- The derivative of a difference is the difference of the derivatives
- Constants can be pulled out front

Example 4. Find the derivative of $f(x) = 2x^5 + 7$.

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^5 + 7) \\ &= \frac{d}{dx}(2x^5) + \frac{d}{dx}(7) \quad (\text{sum rule}) \\ &= 2 \frac{d}{dx}(x^5) + 0 \quad (\text{constant multiple \& constant rules}) \\ &= 2(5x^4) \quad (\text{power rule}) \\ &= 10x^4 \end{aligned}$$

2.2 Practice Problem

Work this out: Find $f'(x)$ for $f(x) = 2x^3 - 6x^2 + 3$.

2.3 Finding Tangent Lines (Revisited)

Now we can find tangent lines much faster!

Example 5. Find the equation of the tangent line to $f(x) = x^2 - 4x + 6$ at $x = 1$.

Solution:

$f'(x) = 2x - 4$, so slope is $f'(1) = -2$.

Point: $f(1) = 3$, so $(1, 3)$.

Equation: $y - 3 = -2(x - 1)$, or $y = -2x + 5$

3 The Product Rule

Warning! The derivative of a product is *NOT* the product of the derivatives!

For example, if $f(x) = x^2 = x \cdot x$, then $f'(x) = 2x$, not $(1)(1) = 1$.

Theorem 4 (Product Rule). If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

In words: (derivative of first)(second) + (first)(derivative of second)

Example 6. Find the derivative of $j(x) = (x^2 + 2)(3x^3 - 5x)$.

Solution:

Let $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$.

Then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$.

Using the product rule:

$$\begin{aligned}j'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (2x)(3x^3 - 5x) + (x^2 + 2)(9x^2 - 5) \\ &= 6x^4 - 10x^2 + 9x^4 - 5x^2 + 18x^2 - 10 \\ &= 15x^4 + 3x^2 - 10\end{aligned}$$

Example 7. For $j(x) = f(x)g(x)$, find $j'(2)$ if $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 6$.

Solution:

$$\begin{aligned}j'(2) &= f'(2)g(2) + f(2)g'(2) \\ &= (-4)(1) + (3)(6) \\ &= -4 + 18 = 14\end{aligned}$$

3.1 Practice Problem

Work this out: Use the product rule to find the derivative of $h(x) = (x^3 + 1)(2x^2 - 3)$.

4 The Quotient Rule

Warning! The derivative of a quotient is *NOT* the quotient of the derivatives!

Theorem 5 (Quotient Rule). If $f(x)$ and $g(x)$ are differentiable and $g(x) \neq 0$, then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

Memory aid: "lo d-hi minus hi d-lo over lo-lo"
where "hi" = numerator, "lo" = denominator, "d" = derivative

Example 8. Find the derivative of $k(x) = \frac{5x^2}{4x + 3}$.

Solution:

Let $f(x) = 5x^2$ (so $f'(x) = 10x$) and $g(x) = 4x + 3$ (so $g'(x) = 4$).

$$\begin{aligned}
k'(x) &= \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} \\
&= \frac{(10x)(4x + 3) - (4)(5x^2)}{(4x + 3)^2} \\
&= \frac{40x^2 + 30x - 20x^2}{(4x + 3)^2} \\
&= \frac{20x^2 + 30x}{(4x + 3)^2}
\end{aligned}$$

Example 9. Find the derivative of $h(x) = \frac{3x + 1}{4x - 3}$.

Solution:

$f(x) = 3x + 1$ (so $f'(x) = 3$), $g(x) = 4x - 3$ (so $g'(x) = 4$)

$$h'(x) = \frac{(3)(4x - 3) - (4)(3x + 1)}{(4x - 3)^2} = \frac{12x - 9 - 12x - 4}{(4x - 3)^2} = \frac{-13}{(4x - 3)^2}$$

4.1 Practice Problem

Work this out: Find the derivative of $f(x) = \frac{x^2 + 1}{x - 2}$.

5 Extended Power Rule

The power rule also works for negative exponents!

Theorem 6 (Extended Power Rule). *If k is a negative integer, then*

$$\frac{d}{dx}(x^k) = kx^{k-1}$$

Example 10. Find $\frac{d}{dx}(x^{-4})$.

Solution:

$$\frac{d}{dx}(x^{-4}) = -4x^{-5}$$

Example 11. Find the derivative of $f(x) = \frac{6}{x^2}$.

Solution:

Rewrite as $f(x) = 6x^{-2}$:

$$\begin{aligned}f'(x) &= 6 \cdot \frac{d}{dx}(x^{-2}) \\ &= 6(-2x^{-3}) \\ &= -12x^{-3} = -\frac{12}{x^3}\end{aligned}$$

Tip: Often it's easier to rewrite quotients using negative exponents before differentiating, rather than using the quotient rule!

6 Combining All the Rules

Most problems require using multiple rules together.

Example 12. Find the derivative of $f(x) = 3x^4 - \frac{2}{x^3} + 5x - 7$.

Solution:

Rewrite as $f(x) = 3x^4 - 2x^{-3} + 5x - 7$:

$$f'(x) = 12x^3 + 6x^{-4} + 5 = 12x^3 + \frac{6}{x^4} + 5$$

Example 13. For $k(x) = 3h(x) + x^2g(x)$, find $k'(x)$.

Solution:

$$k'(x) = 3h'(x) + [2xg(x) + x^2g'(x)] = 3h'(x) + 2xg(x) + x^2g'(x)$$

7 Applications

7.1 Finding Horizontal Tangents

Horizontal tangent lines occur where $f'(x) = 0$.

Example 14. Find where $f(x) = x^3 - 7x^2 + 8x + 1$ has horizontal tangent lines.

Solution:

First, find $f'(x) = 3x^2 - 14x + 8$.

Set $f'(x) = 0$:

$$\begin{aligned}3x^2 - 14x + 8 &= 0 \\ (3x - 2)(x - 4) &= 0 \\ x &= \frac{2}{3} \text{ or } x = 4\end{aligned}$$

The function has horizontal tangent lines at $x = \frac{2}{3}$ and $x = 4$.

7.2 Finding Velocity

Remember: if $s(t)$ is position, then $v(t) = s'(t)$ is velocity.

Example 15. An object's position is $s(t) = \frac{t}{t^2 + 1}$ meters. Find the initial velocity.

Solution:

Use the quotient rule:

$$\begin{aligned} s'(t) &= \frac{(1)(t^2 + 1) - (2t)(t)}{(t^2 + 1)^2} \\ &= \frac{t^2 + 1 - 2t^2}{(t^2 + 1)^2} \\ &= \frac{1 - t^2}{(t^2 + 1)^2} \end{aligned}$$

$$\text{Initial velocity: } v(0) = s'(0) = \frac{1 - 0}{(1)^2} = 1 \text{ m/s}$$

7.3 Practice Problem

Work this out: Find the values of x where $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to $y = 2x + 3$.

Hint: Parallel lines have the same slope!

8 Summary

Basic Rules:

- Constant Rule: $\frac{d}{dx}(c) = 0$
- Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$ (works for negative integers too!)
- Constant Multiple: $\frac{d}{dx}[cf(x)] = cf'(x)$
- Sum/Difference: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Advanced Rules:

- Product Rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

- Quotient Rule: $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$

Strategy: Apply rules in reverse order of how you'd evaluate the function!