Jan15 Lecture

Transformations and Trigonometric Functions

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Transformations of Functions

Transformation of f $(c > 0)$	**Effect on the graph of f**
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $0 < c < 1$; horizontal compression if $c > 1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis

Table 1: Transformations of Functions

1 Introduction & Motivation (10 Minutes)

The Math of Cycles

Trigonometric functions model repetitive or cyclical motion. While you may have first encountered sine, cosine, and tangent in the context of right triangles, these functions have far more power when viewed as functions that describe periodic behavior. Any phenomenon that repeats itself over time or space can often be modeled using trigonometric functions.

Real-World Applications

- **Physics:** Sound waves propagate as sine waves through air. When you pluck a guitar string, it vibrates in a pattern described by trigonometric functions. A pendulum swinging back and forth traces out sinusoidal motion over time.
- **Engineering:** Alternating current (AC) in electrical systems follows a sinusoidal pattern. The voltage in your wall outlet varies as $V(t) = V_0 \sin(\omega t)$, where ω is the angular frequency (typically 60 Hz in the US).
- **Nature:** The number of daylight hours throughout the year follows a sinusoidal pattern, with maximum daylight at the summer solstice and minimum at the winter solstice. Ocean tides rise and fall in patterns that can be modeled using combinations of sine and cosine functions.

Goal

Our goal is to move beyond the right-triangle definitions of trigonometry and understand these functions as continuous, periodic functions defined for all real numbers. This perspective will be essential for calculus and for modeling real-world phenomena.

2 Radian Measure (15 Minutes)

Definition

On a unit circle (radius 1), the radian measure of an angle θ is equal to the arc length s. In other words, if you walk along the circumference of a unit circle and travel a distance of s units, you will have swept out an angle of s radians.

Why Radians?

They provide a more natural measurement for calculus because they link angle to length directly. When working with derivatives and integrals of trigonometric functions, using radians eliminates awkward conversion constants. For instance, the derivative of $\sin(x)$ is $\cos(x)$ only when x is measured in radians.

Additionally, radians give us the beautiful relationship that for small angles θ (in radians), $\sin(\theta) \approx \theta$ and $\tan(\theta) \approx \theta$. This approximation is crucial in physics and engineering.

Conversions

Since the circumference of a unit circle is 2π, we have $360° = 2\pi$ rad, which means $180° = \pi$ rad.

- **Degrees to Radians:** Multiply by $\pi/180°$.
- **Radians to Degrees:** Multiply by $180°/\pi$.

Examples

Example 1: Express 225° in radians.

$$225° \cdot \left(\frac{\pi}{180°}\right) = \frac{225\pi}{180} = \frac{5\pi}{4} \text{ rad}$$

Example 2: Express $\frac{7\pi}{6}$ radians in degrees.

$$\frac{7\pi}{6} \cdot \left(\frac{180°}{\pi}\right) = \frac{7 \cdot 180°}{6} = 210°$$

Example 3: Common angles you should memorize:

Degrees	Radians
30°	$\pi/6$
45°	$\pi/4$
60°	$\pi/3$
90°	$\pi/2$
180°	π
270°	$3\pi/2$
360°	2π

3 The Unit Circle & The Six Functions (15 Minutes)

Standard Position

An angle θ is in standard position when the initial side lies along the positive x-axis and the terminal side rotates counterclockwise (for positive angles) to end at point $P(x, y)$ on the circle. The angle θ is measured from the positive x-axis.

The Definitions

For a point $P(x, y)$ on the unit circle (where $x^2 + y^2 = 1$), we define:

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y} \quad (y \neq 0)$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x} \quad (x \neq 0)$$

$$\tan\theta = \frac{y}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

Notice that these definitions extend the trigonometric functions beyond acute angles. We can now evaluate $\sin(\theta)$ and $\cos(\theta)$ for any angle, positive or negative.

General Circles

For a circle with radius r, if the terminal side of angle θ intersects the circle at (x, y), then:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

This is the foundation for converting between polar and Cartesian coordinates.

Examples

Example 1: Find $\sin(\pi/6)$, $\cos(\pi/6)$, and $\tan(\pi/6)$.

The angle $\pi/6$ radians (or $30°$) corresponds to the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ on the unit circle. Therefore:

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \tan\left(\frac{\pi}{6}\right) = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Example 2: Find $\sin(7\pi/6)$.

The angle $7\pi/6$ is in the third quadrant (since $\pi < 7\pi/6 < 3\pi/2$). Its reference angle is $\pi/6$. In the third quadrant, both x and y are negative, so:

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

4 Trigonometric Identities (10 Minutes)

Trigonometric identities are equations involving trigonometric functions that are true for all values of the variables for which both sides are defined. These identities are essential tools for simplifying expressions and solving equations.

Reciprocal Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta}$$

Similarly, $\sec\theta = \frac{1}{\cos\theta}$ and $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$.

Pythagorean Identities

These come directly from the equation of the unit circle $x^2 + y^2 = 1$:

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The last two can be derived from the first by dividing through by $\cos^2\theta$ or $\sin^2\theta$ respectively.

Double-Angle Formulas

These formulas express trigonometric functions of 2θ in terms of functions of θ:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

The three forms of the cosine double-angle formula are all equivalent and useful in different contexts.

Example

Simplify $\frac{\sin(2\theta)}{1+\cos(2\theta)}$.

Using the double-angle formulas:

$$\frac{\sin(2\theta)}{1+\cos(2\theta)} = \frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta - 1)} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

5 Graphs & Transformations (15 Minutes)

Periodicity

A function is periodic if there exists a positive number P such that $f(x+P) = f(x)$ for all x in the domain. The smallest such P is called the period of the function.

- $\sin, \cos, \csc, \sec$ have a period of 2π. This means their graphs repeat every 2π units.
- $\tan, \cot$ have a period of π. These functions repeat twice as often.

The Master Equation

The general form of a sinusoidal function is:

$$f(x) = A\sin(B(x-\alpha)) + C$$

Each parameter controls a specific transformation:

- $|A|$ **(Amplitude):** The amplitude is the distance from the center line to the maximum (or minimum) of the function. It represents a vertical stretch by a factor of $|A|$. If $A < 0$, the graph is also reflected across the x-axis.
- $\frac{2\pi}{|B|}$ **(Period):** The period tells us how long it takes for the function to complete one full cycle. As B increases, the function oscillates more rapidly (horizontal compression). As B decreases, the oscillations slow down (horizontal stretch).
- α **(Phase Shift):** This is a horizontal shift. If $\alpha > 0$, the graph shifts to the right by α units. If $\alpha < 0$, it shifts left. Be careful: in the form $\sin(B(x-\alpha))$, a positive α shifts right.
- C **(Vertical Shift):** This shifts the entire graph up (if $C > 0$) or down (if $C < 0$). The center line of the oscillation is at $y = C$.

Examples

Example 1: Identify the amplitude, period, and phase shift of $y = 3\sin(2(x - \pi/4)) + 1$.

- Amplitude: $|A| = 3$
- Period: $\frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$
- Phase shift: $\alpha = \pi/4$ (shift right by $\pi/4$)
- Vertical shift: $C = 1$ (center line at $y = 1$)

Example 2: Write an equation for a cosine function with amplitude 4, period π, phase shift $\pi/2$ to the left, and vertical shift down 2 units.

We need $|A| = 4$, so $A = \pm 4$. We'll use $A = 4$.

For period: $\frac{2\pi}{|B|} = \pi \Rightarrow |B| = 2$, so $B = 2$.

Phase shift $\pi/2$ to the left means $\alpha = -\pi/2$.

Vertical shift down 2 means $C = -2$.

Therefore: $y = 4\cos(2(x + \pi/2)) - 2$ or equivalently $y = 4\cos(2x + \pi) - 2$.

Example 3: Analyze the function $y = 3\sin(2(x - \pi/4)) + 1$ and identify all transformations.

First, let's identify each parameter by comparing to the master equation $f(x) = A\sin(B(x-\alpha)) + C$:

- $A = 3$
- $B = 2$
- $\alpha = \pi/4$
- $C = 1$

Now we can determine all the transformations:

Amplitude: $|A| = 3$. The graph oscillates 3 units above and below the center line.

Period: $\frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$. The function completes one full cycle every π units (instead of the standard 2π for basic sine).

Phase Shift: $\alpha = \pi/4$ units to the right. The entire graph is shifted $\pi/4$ units in the positive x-direction.

Vertical Shift: $C = 1$. The center line (midline) of the oscillation is at $y = 1$ instead of $y = 0$.

Key Features:

- Maximum value: $1 + 3 = 4$
- Minimum value: $1 - 3 = -2$
- The function starts its cycle at $x = \pi/4$ (where a basic sine starts at $x = 0$)
- The function completes its first full period at $x = \pi/4 + \pi = 5\pi/4$

6 Challenge Problems & Solutions (10 Minutes)

Challenge 1

Model daylight hours if the longest day is 15.7 hours and the shortest is 8.3.

Answer: Average (Vertical Shift) is $(15.7 + 8.3)/2 = 12$. Amplitude is $15.7 - 12 = 3.7$.
Equation: $h(t) = 3.7\sin(B(t - \alpha)) + 12$.

Challenge 2

Find all solutions to $2\sin\theta - 1 = 0$ on $[0, 2\pi)$.

Answer: $\sin\theta = \frac{1}{2}$. Solutions are $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

Challenge 3

Use an identity to solve $1 + \cos(2\theta) = \cos\theta$.

Answer: Substitute $\cos(2\theta) = 2\cos^2\theta - 1$. This becomes

$$1 + (2\cos^2\theta - 1) = \cos\theta$$

which simplifies to

$$2\cos^2\theta - \cos\theta = 0$$

Factor to find $\cos\theta(2\cos\theta - 1) = 0$.

Case 1: $\cos\theta = 0$
On the interval $[0, 2\pi)$, the cosine is zero at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.
Case 2: $2\cos\theta - 1 = 0 \implies \cos\theta = \frac{1}{2}$
On the interval $[0, 2\pi)$, the cosine is $\frac{1}{2}$ at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.
Final Solution:

$$\theta = \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}\right\}$$