

Calculus Lecture Notes

Introduction to Limits and Their Properties

1 Preview of Calculus

1.1 Motivation: Why Limits?

Calculus arose from practical problems in physics and engineering. Two fundamental problems led to its development:

1. **The Tangent Problem:** How to determine the slope of a line tangent to a curve at a point
2. **The Area Problem:** How to determine the area under a curve

1.2 The Tangent Problem and Rate of Change

1.2.1 Linear Functions

For linear functions, the rate of change is **constant** and determined by the slope:

- $f(x) = -2x - 3$ has slope -2 (decreases 2 units for every 1 unit right)
- $g(x) = \frac{1}{2}x + 1$ has slope $\frac{1}{2}$ (increases $\frac{1}{2}$ unit for every 1 unit right)
- $h(x) = 2$ has slope 0 (constant function, no change)

1.2.2 Nonlinear Functions

For $k(x) = x^2$, the rate of change is **not constant**. The graph:

- Decreases rapidly on the left
- Levels off near the origin
- Increases slowly, then more rapidly on the right

Question: How do we measure the rate of change for nonlinear functions?

Definition 1.1 (Secant Line). The **secant line** to the function $f(x)$ through points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is:

$$m_{sec} = \frac{f(x) - f(a)}{x - a} \quad (1)$$

Key Insight: As x gets closer to a , the secant line becomes a better approximation of the rate of change at a .

Definition 1.2 (Tangent Line). The **tangent line** to $f(x)$ at a is the line that the secant lines approach as x approaches a . The slope of the tangent line is called the **derivative** of f at a , denoted $f'(a)$.

Example 1.1 (Finding Slopes of Secant Lines). Estimate the slope of the tangent line to $f(x) = x^2$ at $x = 1$ using secant lines through $(1, 1)$ and:

(a) $(2, 4)$:

$$m_{sec} = \frac{4 - 1}{2 - 1} = 3$$

(b) $(\frac{3}{2}, \frac{9}{4})$:

$$m_{sec} = \frac{\frac{9}{4} - 1}{\frac{3}{2} - 1} = \frac{\frac{5}{4}}{\frac{1}{2}} = 2.5$$

Since point (b) is closer to $(1, 1)$, the slope 2.5 is a better estimate. The actual slope of the tangent line is between 2 and 2.5.

1.3 Instantaneous Velocity

Definition 1.3 (Average Velocity). For a position function $s(t)$, the **average velocity** over time interval $[a, t]$ is:

$$v_{ave} = \frac{s(t) - s(a)}{t - a} \quad (2)$$

Definition 1.4 (Instantaneous Velocity). The **instantaneous velocity** at time $t = a$ is the value that average velocities approach as t gets closer to a .

Example 1.2 (Average Velocity of Falling Rock). A rock dropped from 64 ft has height $s(t) = -16t^2 + 64$ for $0 \leq t \leq 2$. Find average velocity over intervals near $t = 0.5$:

(a) $[0.49, 0.5]$:

$$v_{ave} = \frac{s(0.5) - s(0.49)}{0.5 - 0.49} = -15.84 \text{ ft/sec}$$

(b) $[0.5, 0.51]$:

$$v_{ave} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.16 \text{ ft/sec}$$

The instantaneous velocity at $t = 0.5$ is approximately -16 ft/sec.

1.4 The Area Problem

To find the area between the graph of $f(x)$ and the x -axis over $[a, b]$:

1. Divide $[a, b]$ into smaller intervals
2. Approximate with rectangles
3. Sum the areas of rectangles
4. As rectangle widths approach zero, the sum approaches the actual area

This limiting process leads to the **definite integral**, the foundation of integral calculus.

2 The Limit of a Function

2.1 Intuitive Definition

Consider three functions undefined at $x = 2$:

$$f(x) = \frac{x^2 - 4}{x - 2}, \quad g(x) = \frac{|x - 2|}{x - 2}, \quad h(x) = \frac{1}{(x - 2)^2}$$

Each behaves differently near $x = 2$.

Definition 2.1 (Limit of a Function - Intuitive). *Let $f(x)$ be defined on an open interval containing a (except possibly at a itself). If all values of $f(x)$ approach a real number L as x approaches a , we write:*

$$\lim_{x \rightarrow a} f(x) = L \tag{3}$$

We say “the limit of $f(x)$ as x approaches a is L .”

2.2 Evaluating Limits Using Tables

Strategy:

1. Create two columns: values approaching a from the left and right
2. Use x -values like $a \pm 0.1, a \pm 0.01, a \pm 0.001$, etc.
3. Calculate corresponding $f(x)$ values
4. Check if both columns approach the same value L

Example 2.1 (Limit Using Table). *Evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ using a table:*

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
-0.1	0.998334	0.1	0.998334
-0.01	0.999983	0.01	0.999983
-0.001	0.999999	0.001	0.999999

Both columns approach 1, so $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

2.3 Two Important Limits

Theorem 2.1 (Basic Limits). *For any real number a and constant c :*

$$\lim_{x \rightarrow a} x = a \quad (4)$$

$$\lim_{x \rightarrow a} c = c \quad (5)$$

2.4 When Limits Don't Exist

Example 2.2 (Limit That Does Not Exist). *Evaluate $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$.*

Using the sequence $x = \frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \dots$ approaching 0:

The corresponding y -values are: $1, -1, 1, -1, 1, -1, \dots$

Since the function oscillates between -1 and 1 without approaching a single value:

$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) \text{ DNE (does not exist)}$$

2.5 One-Sided Limits

Definition 2.2 (One-Sided Limits). **Left-hand limit:** *If $f(x)$ approaches L as x approaches a from the left ($x < a$):*

$$\lim_{x \rightarrow a^-} f(x) = L \quad (6)$$

Right-hand limit: *If $f(x)$ approaches L as x approaches a from the right ($x > a$):*

$$\lim_{x \rightarrow a^+} f(x) = L \quad (7)$$

Example 2.3 (One-Sided Limits). *For $f(x) = \begin{cases} x + 1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$, evaluate limits at $x = 2$:*

From the left: *As $x \rightarrow 2^-$, use $f(x) = x + 1$:*

$$\lim_{x \rightarrow 2^-} f(x) = 2 + 1 = 3$$

From the right: *As $x \rightarrow 2^+$, use $f(x) = x^2 - 4$:*

$$\lim_{x \rightarrow 2^+} f(x) = 2^2 - 4 = 0$$

Since the one-sided limits differ, $\lim_{x \rightarrow 2} f(x)$ does not exist.

Theorem 2.2 (Relationship Between One-Sided and Two-Sided Limits). *$\lim_{x \rightarrow a} f(x) = L$ if and only if:*

$$\lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L$$

2.6 Infinite Limits

Definition 2.3 (Infinite Limits). We write $\lim_{x \rightarrow a} f(x) = +\infty$ if $f(x)$ increases without bound as $x \rightarrow a$.

We write $\lim_{x \rightarrow a} f(x) = -\infty$ if $f(x)$ decreases without bound as $x \rightarrow a$.

Note: These limits do not exist in the traditional sense, but the notation describes the behavior.

Example 2.4 (Infinite Limits). Evaluate $\lim_{x \rightarrow 0} \frac{1}{x}$:

From the left: As $x \rightarrow 0^-$, values are negative and $\frac{1}{x} \rightarrow -\infty$

From the right: As $x \rightarrow 0^+$, values are positive and $\frac{1}{x} \rightarrow +\infty$

Since the one-sided limits differ:

$$\lim_{x \rightarrow 0} \frac{1}{x} \text{ DNE}$$

Theorem 2.3 (Infinite Limits from Positive Integers). For positive integer n :

If n is even:

$$\lim_{x \rightarrow a} \frac{1}{(x-a)^n} = +\infty$$

If n is odd:

$$\lim_{x \rightarrow a^+} \frac{1}{(x-a)^n} = +\infty \quad \text{and} \quad \lim_{x \rightarrow a^-} \frac{1}{(x-a)^n} = -\infty$$

Definition 2.4 (Vertical Asymptote). The line $x = a$ is a **vertical asymptote** of $f(x)$ if any of the following hold:

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty$$

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty$$

$$\lim_{x \rightarrow a} f(x) = \pm\infty$$

Example 2.5 (Finding Vertical Asymptotes). Find vertical asymptotes of $f(x) = \frac{1}{(x+3)^4}$:

Since the exponent is even:

$$\lim_{x \rightarrow -3} \frac{1}{(x+3)^4} = +\infty$$

Therefore, $x = -3$ is a vertical asymptote.

3 Key Takeaways

1. **Limits** describe the behavior of functions near a point
2. The limit of $f(x)$ at $x = a$ may exist even if $f(a)$ is undefined
3. The limit and the function value can be different: $\lim_{x \rightarrow a} f(x) \neq f(a)$

4. For a limit to exist, left-hand and right-hand limits must be equal
5. Infinite limits describe unbounded behavior (vertical asymptotes)
6. Limits are essential for defining derivatives and integrals

4 Practice Problems

1. Estimate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$ using a table of values.
2. For $g(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$, find $\lim_{x \rightarrow 1^-} g(x)$ and $\lim_{x \rightarrow 1^+} g(x)$.
3. Find all vertical asymptotes of $h(x) = \frac{1}{(x-2)^3}$.
4. Does $\lim_{x \rightarrow 0} \frac{|x|}{x}$ exist? Explain.

Answers to Practice Problems

1. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6$

Solution: Factor the numerator: $\frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$.

Create a table:

x	$\frac{x^2-9}{x-3}$	x	$\frac{x^2-9}{x-3}$
2.9	5.9	3.1	6.1
2.99	5.99	3.01	6.01
2.999	5.999	3.001	6.001

Both columns approach 6, so $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6$.

2. $\lim_{x \rightarrow 1^-} g(x) = 2$ and $\lim_{x \rightarrow 1^+} g(x) = 1$

Solution:

From the left ($x < 1$): Use $g(x) = 2x$, so $\lim_{x \rightarrow 1^-} g(x) = 2(1) = 2$

From the right ($x \geq 1$): Use $g(x) = x^2$, so $\lim_{x \rightarrow 1^+} g(x) = 1^2 = 1$

Since the one-sided limits are different, $\lim_{x \rightarrow 1} g(x)$ does not exist.

3. The vertical asymptote is at $x = 2$

Solution: The function $h(x) = \frac{1}{(x-2)^3}$ has a denominator that equals zero when $x = 2$.

Since the exponent is odd (3): $\lim_{x \rightarrow 2^+} \frac{1}{(x-2)^3} = +\infty$ $\lim_{x \rightarrow 2^-} \frac{1}{(x-2)^3} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

4. No, the limit does not exist.

Solution: We need to check one-sided limits.

Recall that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

From the left ($x < 0$): $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} (-1) = -1$

From the right ($x > 0$): $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$

Since $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1 \neq 1 = \lim_{x \rightarrow 0^+} \frac{|x|}{x}$, the two-sided limit does not exist.