Calculus Lecture Notes

The Limit Laws

1 Introduction

In previous sections, we evaluated limits by:

- Looking at graphs
- Constructing tables of values

In this section, we establish **laws for calculating limits** and learn how to apply them systematically.

2 Basic Limit Results

Theorem 2.1 (Basic Limit Results). *For any real number a and any constant c:*

$$\lim_{x \to a} x = a \tag{1}$$

$$\lim_{x \to a} c = c \tag{2}$$

Example 2.1 (Evaluating Basic Limits). *Evaluate each limit:*

(a) $\lim_{x \to 2} x$

Solution: *The limit of x as x approaches a is a:*

$$\lim_{x \to 2} x = 2$$

(b) $\lim_{x \to 2} 5$

Solution: *The limit of a constant is that constant:*

$$\lim_{x \to 2} 5 = 5$$

3 The Limit Laws

Theorem 3.1 (Limit Laws). *Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, where L and M are real numbers. Let c be a constant. Then:*

1. ***Sum Law:***
$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$

2. ***Difference Law:***
$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$$

3. ***Constant Multiple Law:***
$$\lim_{x\to a} cf(x) = c\cdot \lim_{x\to a} f(x) = cL$$

4. ***Product Law:***
$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = L\cdot M$$

5. ***Quotient Law:***
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \quad \textit{for } M \neq 0$$

6. ***Power Law:***
$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n \quad \textit{for every positive integer } n$$

7. ***Root Law:***
$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$$
for all L if n is odd, and for $L \geq 0$ if n is even

Example 3.1 (Using Limit Laws Step by Step)**.** *Use the limit laws to evaluate* $\lim_{x\to -3}(4x+2)$.

Solution: *Apply the limit laws one step at a time:*

$$\begin{aligned}
\lim_{x\to -3}(4x+2) &= \lim_{x\to -3} 4x + \lim_{x\to -3} 2 && \textit{(Sum law)}\\
&= 4\cdot \lim_{x\to -3} x + \lim_{x\to -3} 2 && \textit{(Constant multiple law)}\\
&= 4\cdot(-3)+2 && \textit{(Basic limit results)}\\
&= -12+2 = -10
\end{aligned}$$

Example 3.2 (Using Limit Laws Repeatedly)**.** *Use the limit laws to evaluate* $\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}$.

Solution: *Apply several limit laws:*

$$\begin{aligned}
\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4} &= \frac{\lim_{x\to 2}(2x^2-3x+1)}{\lim_{x\to 2}(x^3+4)} && \textit{(Quotient law, } 2^3+4\neq 0\textit{)}\\
&= \frac{2\cdot \lim_{x\to 2} x^2 - 3\cdot \lim_{x\to 2} x + \lim_{x\to 2} 1}{\lim_{x\to 2} x^3 + \lim_{x\to 2} 4} && \textit{(Sum, constant multiple)}\\
&= \frac{2\cdot (\lim_{x\to 2} x)^2 - 3\cdot \lim_{x\to 2} x + \lim_{x\to 2} 1}{(\lim_{x\to 2} x)^3 + \lim_{x\to 2} 4} && \textit{(Power law)}\\
&= \frac{2(4)-3(2)+1}{(2)^3+4} && \textit{(Basic limits)}\\
&= \frac{8-6+1}{8+4} = \frac{3}{12} = \frac{1}{4}
\end{aligned}$$

4 Limits of Polynomial and Rational Functions

Theorem 4.1 (Limits of Polynomial and Rational Functions)**.** *Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then:*

$$\lim_{x \to a} p(x) = p(a)$$
$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \quad \text{when } q(a) \neq 0$$

Key Idea: For polynomials and rational functions (where the denominator is non-zero), we can evaluate limits by **direct substitution**.

Example 4.1 (Limit of a Rational Function)**.** *Evaluate* $\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

Solution: *Since 3 is in the domain of the rational function (the denominator $5(3) + 4 = 19 \neq 0$), we use direct substitution:*

$$\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{2(3)^2 - 3(3) + 1}{5(3) + 4} = \frac{18 - 9 + 1}{15 + 4} = \frac{10}{19}$$

5 The Indeterminate Form 0/0

When direct substitution gives 0/0, we have an **indeterminate form**. The limit may exist, but we need additional techniques.

5.1 Strategy for Evaluating Limits with Form 0/0

1. Verify the limit has the form 0/0 (cannot be evaluated immediately)
2. Find a function equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ by:
   - **Factoring** polynomials and canceling common factors
   - **Multiplying by a conjugate** if there's a square root
   - **Simplifying** complex fractions
3. Apply the limit laws to the simplified function

5.2 Technique 1: Factoring and Canceling

Example 5.1 (Evaluating by Factoring)**.** *Evaluate* $\lim_{x \to 3} \frac{x^2 - 3x}{2x^2 - 5x - 3}$.

Solution:

Step 1: *Check the form. Substituting $x = 3$:*

$$\frac{3^2 - 3(3)}{2(3)^2 - 5(3) - 3} = \frac{9 - 9}{18 - 15 - 3} = \frac{0}{0}$$

Step 2: *Factor and cancel:*

$$\begin{aligned}\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3} &= \lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}\\ &= \lim_{x\to 3}\frac{x}{2x+1} &&\textit{(for } x\neq 3\textit{)}\end{aligned}$$

Step 3: *Apply limit laws:*

$$\lim_{x\to 3}\frac{x}{2x+1}=\frac{3}{2(3)+1}=\frac{3}{7}$$

5.3 Technique 2: Multiplying by a Conjugate

Example 5.2 (Evaluating by Conjugate)**.** *Evaluate* $\lim_{x\to -1}\dfrac{\sqrt{x+2}-1}{x+1}$.

Solution:

Step 1: *Check:* $\frac{\sqrt{-1+2}-1}{-1+1}=\frac{1-1}{0}=\frac{0}{0}$

Step 2: *Multiply by the conjugate* $\sqrt{x+2}+1$*:*

$$\begin{aligned}\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1} &= \lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}\\ &= \lim_{x\to -1}\frac{(x+2)-1}{(x+1)(\sqrt{x+2}+1)}\\ &= \lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+2}+1)}\end{aligned}$$

Step 3: *Cancel and evaluate:*

$$=\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

5.4 Technique 3: Simplifying Complex Fractions

Example 5.3 (Simplifying a Complex Fraction)**.** *Evaluate* $\lim_{x\to 1}\dfrac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$.

Solution:

Step 1: *Check:* $\frac{\frac{1}{2}-\frac{1}{2}}{0}=\frac{0}{0}$

Step 2: *Multiply by* $\frac{2(x+1)}{2(x+1)}$*:*

$$\begin{aligned}\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} &= \lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}\cdot\frac{2(x+1)}{2(x+1)}\\ &= \lim_{x\to 1}\frac{2-(x+1)}{2(x-1)(x+1)}\end{aligned}$$

Step 3: *Simplify the numerator:*

$$=\lim_{x\to 1}\frac{2-x-1}{2(x-1)(x+1)}=\lim_{x\to 1}\frac{-x+1}{2(x-1)(x+1)}$$

Step 4: *Factor out -1:*

$$= \lim_{x\to 1} \frac{-(x-1)}{2(x-1)(x+1)}$$

Step 5: *Cancel and evaluate:*

$$= \lim_{x\to 1} \frac{-1}{2(x+1)} = \frac{-1}{2(2)} = -\frac{1}{4}$$

6 One-Sided Limits and Limit Laws

The limit laws apply to one-sided limits with appropriate domain restrictions:

- For $\lim_{x\to a^-} h(x)$: $h(x)$ must be defined on an interval (b, a)
- For $\lim_{x\to a^+} h(x)$: $h(x)$ must be defined on an interval (a, c)

Example 6.1 (One-Sided Limits)**.** *Evaluate the limits for $f(x) = \sqrt{x-3}$:*
(a) $\lim\limits_{x\to 3^-} \sqrt{x-3}$
Solution: *The function $f(x) = \sqrt{x-3}$ is defined only for $x \geq 3$. It is not defined to the left of 3, so:*

$$\lim_{x\to 3^-} \sqrt{x-3} \textit{ does not exist}$$

(b) $\lim\limits_{x\to 3^+} \sqrt{x-3}$
Solution: *The function is defined to the right of 3, so we can apply limit laws:*

$$\lim_{x\to 3^+} \sqrt{x-3} = \sqrt{\lim_{x\to 3^+} (x-3)} = \sqrt{0} = 0$$

Example 6.2 (Piecewise Function Limits)**.** *For $f(x) = \begin{cases} 4x-3 & \textit{if } x < 2 \\ (x-3)^2 & \textit{if } x \geq 2 \end{cases}$, evaluate:*

(a) $\lim\limits_{x\to 2^-} f(x)$
Solution: *For $x < 2$, use $f(x) = 4x - 3$:*

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (4x-3) = 4(2) - 3 = 5$$

(b) $\lim\limits_{x\to 2^+} f(x)$
Solution: *For $x \geq 2$, use $f(x) = (x-3)^2$:*

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-3)^2 = (2-3)^2 = 1$$

(c) $\lim\limits_{x\to 2} f(x)$
Solution: *Since $\lim_{x\to 2^-} f(x) = 5 \neq 1 = \lim_{x\to 2^+} f(x)$:*

$$\lim_{x\to 2} f(x) \textit{ does not exist}$$

7 Limits of the Form $K/0$

When $\lim_{x\to a} f(x) = K$ (where $K \neq 0$) and $\lim_{x\to a} g(x) = 0$, the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is either $+\infty$ or $-\infty$.

Example 7.1 (Limit of Form $K/0$)**.** *Evaluate* $\lim_{x\to 2^-} \dfrac{x-3}{x^2-2x}$.

Solution:
Step 1: *Substitute* $x = 2$*:* $\frac{2-3}{2^2-2(2)} = \frac{-1}{0}$ *(form* $K/0$*)*
Step 2: *Factor the denominator:*

$$\lim_{x\to 2^-} \frac{x-3}{x^2-2x} = \lim_{x\to 2^-} \frac{x-3}{x(x-2)}$$

Step 3: *Separate the parts:*

$$= \lim_{x\to 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}$$

Step 4: *Evaluate each part:*

- $\lim_{x\to 2^-} \frac{x-3}{x} = \frac{2-3}{2} = -\frac{1}{2}$
- $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$ *(denominator approaches* 0 *from the left, so it's negative)*

Step 5: *Therefore:*

$$\lim_{x\to 2^-} \frac{x-3}{x^2-2x} = \left(-\frac{1}{2}\right) \cdot (-\infty) = +\infty$$

8 The Squeeze Theorem

Theorem 8.1 (The Squeeze Theorem)**.** *Let* $f(x)$*,* $g(x)$*, and* $h(x)$ *be defined for all* $x \neq a$ *over an open interval containing* a*. If*

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ *in the interval, and*

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

where L *is a real number, then*

$$\lim_{x\to a} g(x) = L$$

Key Idea: If $g(x)$ is "squeezed" between $f(x)$ and $h(x)$, and both f and h approach the same limit L, then g must also approach L.

Example 8.1 (Applying the Squeeze Theorem). *Use the squeeze theorem to evaluate* $\lim_{x \to 0} x \cos x$.

Solution:
Step 1: *We know* $-1 \leq \cos x \leq 1$ *for all* x.
Step 2: *Multiply by* $|x|$: $-|x| \leq x \cos x \leq |x|$
Step 3: *Find the limits of the bounding functions:*

$$\lim_{x \to 0}(-|x|) = 0 \quad \textit{and} \quad \lim_{x \to 0} |x| = 0$$

Step 4: *By the squeeze theorem:*

$$\lim_{x \to 0} x \cos x = 0$$

9 Important Trigonometric Limits

Using the squeeze theorem and geometric arguments, we can establish:

Theorem 9.1 (Important Trigonometric Limits).

$$\lim_{\theta \to 0} \sin \theta = 0 \tag{3}$$

$$\lim_{\theta \to 0} \cos \theta = 1 \tag{4}$$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \tag{5}$$

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0 \tag{6}$$

Note: The angle θ must be in **radians** for these limits to hold.

9.1 Deriving $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Using the unit circle, we can show that for $0 < \theta < \frac{\pi}{2}$:

$$\sin \theta < \theta < \tan \theta$$

Dividing by $\sin \theta > 0$:

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

Taking reciprocals (and reversing inequalities):

$$1 > \frac{\sin \theta}{\theta} > \cos \theta$$

Since $\lim_{\theta \to 0^+} 1 = 1$ and $\lim_{\theta \to 0^+} \cos \theta = 1$, by the squeeze theorem:

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$$

A similar argument shows $\lim_{\theta \to 0^-} \frac{\sin \theta}{\theta} = 1$, so:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Example 9.1 (Important Trigonometric Limit). *Evaluate* $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}$.

Solution: *Multiply by the conjugate:*

$$\begin{aligned}
\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} &= \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} \\
&= \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} && \textit{(identity: } \sin^2\theta = 1-\cos^2\theta\textit{)} \\
&= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1+\cos\theta} \\
&= 1 \cdot \frac{0}{1+1} = 0
\end{aligned}$$

10 Key Takeaways

1. The **limit laws** allow us to break down complex limits into simpler parts
2. For **polynomials** and **rational functions** (where defined), use direct substitution
3. When you get **0/0**, try:
   - Factoring and canceling
   - Multiplying by a conjugate
   - Simplifying complex fractions
4. The **squeeze theorem** is powerful for functions bounded by other functions
5. Important trigonometric limits are essential for calculus (especially derivatives)

11 Practice Problems

1. Use the limit laws to evaluate $\lim_{x\to -2}(3x^3 - 2x + 7)$.

2. Evaluate $\lim_{x\to 4} \frac{x^2-16}{x-4}$ by factoring.

3. Evaluate $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5}$ by multiplying by a conjugate.

4. Evaluate $\lim_{x\to -3} \frac{\frac{1}{x+2}+1}{x+3}$ by simplifying.

5. For $f(x) = \begin{cases} -x-2 & \text{if } x < -1 \\ 2 & \text{if } x = -1 \\ x^3 & \text{if } x > -1 \end{cases}$, find $\lim_{x\to -1^-} f(x)$.

6. Evaluate $\lim_{x\to 1^+} \frac{x+2}{(x-1)^2}$.

7. Use the squeeze theorem to evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

8. Evaluate $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta}$.

Answers to Practice Problems

1. $\lim_{x \to -2} (3x^3 - 2x + 7) = -13$

   Solution: This is a polynomial, so use direct substitution:

   $$\lim_{x \to -2} (3x^3 - 2x + 7) = 3(-2)^3 - 2(-2) + 7 = 3(-8) + 4 + 7 = -24 + 11 = -13$$

2. $\lim_{x \to 4} \dfrac{x^2 - 16}{x - 4} = 8$

   Solution: Direct substitution gives $\frac{0}{0}$, so factor:

   $$\begin{aligned} \lim_{x \to 4} \frac{x^2 - 16}{x - 4} &= \lim_{x \to 4} \frac{(x-4)(x+4)}{x-4} \\ &= \lim_{x \to 4} (x + 4) = 4 + 4 = 8 \end{aligned}$$

3. $\lim_{x \to 5} \dfrac{\sqrt{x-1} - 2}{x - 5} = \dfrac{1}{4}$

   Solution: Multiply by conjugate $\sqrt{x-1} + 2$:

   $$\begin{aligned} \lim_{x \to 5} \frac{\sqrt{x-1} - 2}{x - 5} &= \lim_{x \to 5} \frac{\sqrt{x-1} - 2}{x - 5} \cdot \frac{\sqrt{x-1} + 2}{\sqrt{x-1} + 2} \\ &= \lim_{x \to 5} \frac{(x-1) - 4}{(x-5)(\sqrt{x-1} + 2)} \\ &= \lim_{x \to 5} \frac{x - 5}{(x-5)(\sqrt{x-1} + 2)} \\ &= \lim_{x \to 5} \frac{1}{\sqrt{x-1} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4} \end{aligned}$$

4. $\lim_{x \to -3} \dfrac{\frac{1}{x+2} + 1}{x + 3} = -1$

   Solution: Simplify by multiplying by $\frac{x+2}{x+2}$:

   $$\begin{aligned} \lim_{x \to -3} \frac{\frac{1}{x+2} + 1}{x + 3} &= \lim_{x \to -3} \frac{\frac{1}{x+2} + 1}{x + 3} \cdot \frac{x+2}{x+2} \\ &= \lim_{x \to -3} \frac{1 + (x+2)}{(x+3)(x+2)} \\ &= \lim_{x \to -3} \frac{x + 3}{(x+3)(x+2)} \\ &= \lim_{x \to -3} \frac{1}{x+2} = \frac{1}{-3+2} = \frac{1}{-1} = -1 \end{aligned}$$

5. $\lim_{x \to -1^-} f(x) = -1$

   Solution: For $x < -1$, use $f(x) = -x - 2$: $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-}(-x - 2) = -(-1) - 2 = 1 - 2 = -1$

6. $\lim_{x \to 1^+} \frac{x+2}{(x-1)^2} = +\infty$

   Solution: As $x \to 1^+$:

   - Numerator: $x + 2 \to 1 + 2 = 3$ (positive)
   - Denominator: $(x-1)^2 \to 0^+$ (always positive, approaching 0)

   Therefore: $\lim_{x \to 1^+} \frac{x+2}{(x-1)^2} = +\infty$

7. $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

   Solution: We know $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for all $x \ne 0$.

   Multiply by $x^2 \ge 0$: $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$

   Since $\lim_{x \to 0}(-x^2) = 0$ and $\lim_{x \to 0} x^2 = 0$, by the squeeze theorem: $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

8. $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin\theta} = 0$

   Solution: Multiply by the conjugate:

$$\begin{aligned}
\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin\theta} &= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta} \\
&= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\sin\theta(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2\theta}{\sin\theta(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin\theta}{1 + \cos\theta} = \frac{0}{1+1} = 0
\end{aligned}$$