Continuity
Calculus I - Lecture Notes

January 27, 2026

Motivating Question

Can you draw the graph of a function without lifting your pencil from the paper?

This simple idea captures the intuitive notion of **continuity**. Today we'll make this precise and explore when functions have this property.

1 Introduction: Why Continuity Matters

Consider modeling the temperature throughout a day. Does temperature jump suddenly from 50°F to 75°F? No—it changes smoothly. Functions modeling real-world phenomena are often continuous.

However, some situations involve discontinuities:

- Postage rates (jumps at weight thresholds)
- Light switches (on/off states)
- Tax brackets (marginal rate changes)

Understanding continuity helps us know when we can apply powerful theorems in calculus.

2 Continuity at a Point

2.1 Building the Definition

Let's construct the definition by considering what can go wrong.

Definition 1 (Continuity at a Point)**.** *A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:*

(i) *$f(a)$ is defined*

(ii) $\lim_{x\to a} f(x)$ *exists*

(iii) $\lim_{x\to a} f(x) = f(a)$

A function is **discontinuous at** a if it fails to be continuous at a.

2.2 Checking Continuity: A Three-Step Process

To determine if f is continuous at a:

1. Check if $f(a)$ is defined. If not, **stop**—discontinuous.
2. Compute $\lim_{x\to a} f(x)$. If it doesn't exist, **stop**—discontinuous.
3. Compare $f(a)$ and $\lim_{x\to a} f(x)$. If equal, continuous. If not, discontinuous.

2.3 Worked Examples

Example 1 (Condition 1 Fails)**.** *Determine whether* $f(x) = \frac{x^2-4}{x-2}$ *is continuous at* $x = 2$.

Solution:

$$f(2) = \frac{2^2-4}{2-2} = \frac{0}{0} \quad \textit{(undefined)}$$

Since $f(2)$ *is undefined, condition (i) fails. Therefore,* f *is* ***discontinuous at*** $x = 2$.

Note: *Even though* $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$ *exists, the function is still discontinuous because it's not defined at* $x = 2$.

Example 2 (Condition 2 Fails)**.** *Determine whether* $f(x) = \begin{cases} -x^2+4 & \text{if } x \le 3 \\ 4x-8 & \text{if } x > 3 \end{cases}$ *is continuous at* $x = 3$.

Solution:
Step 1: Is $f(3)$ *defined?*

$$f(3) = -3^2 + 4 = -9 + 4 = -5 \quad \checkmark$$

Step 2: Does $\lim_{x\to 3} f(x)$ *exist?*

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(-x^2+4) = -9+4 = -5$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(4x-8) = 12-8 = 4$$

Since $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$*, the limit* $\lim_{x\to 3} f(x)$ *does not exist.*
Therefore, f *is* ***discontinuous at*** $x = 3$.

Example 3 (All Conditions Satisfied)**.** *Determine whether* $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ *is continuous at* $x = 0$.

Solution:
Step 1: $f(0) = 1$ ✓
Step 2: $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$ ✓
Step 3: $f(0) = 1 = \lim_{x\to 0} f(x)$ ✓
All three conditions are satisfied, so f *is* ***continuous at*** $x = 0$.

2.4 Practice Problem for Class

Work this out: Is $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x+4 & \text{if } x > 1 \end{cases}$ continuous at $x = 1$?

3 Types of Discontinuities

Definition 2 (Classification of Discontinuities)**.** *If* $f(x)$ *is discontinuous at* a*, then:*

1. f *has a* ***removable discontinuity*** *at* a *if* $\lim_{x\to a} f(x)$ *exists (as a finite number).*

2. f *has a* ***jump discontinuity*** *at* a *if* $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a^+} f(x)$ *both exist (as finite numbers), but* $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.

3. f *has an* ***infinite discontinuity*** *at* a *if* $\lim_{x\to a^-} f(x) = \pm\infty$ *or* $\lim_{x\to a^+} f(x) = \pm\infty$.

3.1 Examples of Each Type

Example 4 (Removable Discontinuity)**.** *Classify the discontinuity of* $f(x) = \frac{x^2-4}{x-2}$ *at* $x = 2$.

Solution: *We already know* f *is discontinuous at* $x = 2$*. Now:*

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$$

Since the limit exists (and is finite), this is a ***removable discontinuity****.*
Why "removable"? *If we redefine* $f(2) = 4$*, the function becomes continuous!*

Example 5 (Jump Discontinuity). *The piecewise function from Example 2 has a **jump discontinuity** at $x = 3$ because:*

$$\lim_{x\to 3^-} f(x) = -5 \quad \textit{and} \quad \lim_{x\to 3^+} f(x) = 4$$

Both one-sided limits exist, but they're not equal—the function "jumps" from -5 to 4.

Example 6 (Infinite Discontinuity). *Determine whether $f(x) = \frac{x+2}{x+1}$ is continuous at $x = -1$. If not, classify the discontinuity.*

Solution: *$f(-1)$ is undefined (discontinuous). Now check the limits:*

$$\lim_{x\to -1^-} \frac{x+2}{x+1} = -\infty$$
$$\lim_{x\to -1^+} \frac{x+2}{x+1} = +\infty$$

*This is an **infinite discontinuity** (vertical asymptote at $x = -1$).*

4 Important Theorems

theorem 1 (Continuity of Polynomials and Rational Functions).

- *Polynomials are continuous everywhere (at every real number).*
- *Rational functions are continuous at every point in their domains.*

Example: $f(x) = \frac{x+1}{x-5}$ is continuous for all $x \neq 5$.

theorem 2 (Continuity of Trigonometric Functions). *The functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are continuous at every point in their domains.*

5 Continuity on an Interval

Definition 3 (Continuity on an Interval).

- *A function is **continuous on an open interval** (a, b) if it is continuous at every point in the interval.*
- *A function is **continuous on a closed interval** $[a, b]$ if:*
  - *It is continuous at every point in (a, b)*
  - $\lim_{x\to a^+} f(x) = f(a)$ *(continuous from the right at a)*
  - $\lim_{x\to b^-} f(x) = f(b)$ *(continuous from the left at b)*

Example 7 (Interval of Continuity). *State the intervals over which $f(x) = \frac{x-1}{x^2+2x}$ is continuous.*

Solution: *Since f is a rational function, it's continuous on its domain.*

$$\begin{aligned} x^2 + 2x &= 0 \\ x(x+2) &= 0 \\ x &= 0 \text{ or } x = -2 \end{aligned}$$

Domain: $(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$
Therefore, f is continuous on each interval: $(-\infty, -2)$, $(-2, 0)$, *and* $(0, \infty)$.

6 The Intermediate Value Theorem

theorem 3 (Intermediate Value Theorem (IVT)). *Let f be continuous on a closed interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = z$.*

Intuition: A continuous function must pass through every value between $f(a)$ and $f(b)$—it can't "skip over" any values.

6.1 Application: Proving Zeros Exist

Example 8 (Using IVT to Show a Zero Exists). *Show that $f(x) = x - \cos x$ has at least one zero.*

Solution: *Since $f(x) = x - \cos x$ is continuous everywhere (polynomial minus cosine), we can apply IVT on any closed interval.*
Let's check values:

$$f(0) = 0 - \cos(0) = 0 - 1 = -1 < 0$$
$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2} > 0$$

Since f is continuous on $\left[0, \frac{\pi}{2}\right]$, $f(0) < 0$, and $f\left(\frac{\pi}{2}\right) > 0$, by the IVT there must exist some $c \in \left(0, \frac{\pi}{2}\right)$ such that $f(c) = 0$.
Therefore, $f(x) = x - \cos x$ has at least one zero.

6.2 Practice Problem for Class

Work this out: Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero on the interval $[0, 1]$.

7 Summary

- A function is continuous at a if: (i) $f(a)$ is defined, (ii) $\lim_{x \to a} f(x)$ exists, and (iii) $\lim_{x \to a} f(x) = f(a)$
- Types of discontinuities: removable, jump, infinite
- Polynomials, rational functions, and trig functions are continuous on their domains
- The IVT guarantees that continuous functions on $[a, b]$ take on all intermediate values between $f(a)$ and $f(b)$