The Precise Definition of a Limit
Calculus I - Lecture Notes

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Motivating Question

What does it *really* mean to say that $\lim_{x \to a} f(x) = L$?

So far, we've used intuitive language: "As x gets closer to a, $f(x)$ gets closer to L." But how close is "close"? Today we make this idea mathematically precise using the **epsilon-delta definition**.

1 Why Do We Need a Precise Definition?

Our intuitive understanding of limits has served us well, but:

- Mathematics requires *rigorous* definitions to prove theorems
- "Getting closer" is too vague for formal proofs
- We need to quantify exactly what "arbitrarily close" means
- This definition is the foundation for all of calculus

Think of it like the difference between "it's hot outside" and "the temperature is 95°F." The precise definition gives us a way to measure closeness.

2 Quantifying Closeness

Before we state the formal definition, we need to understand how to measure distance on the number line.

2.1 Distance and Absolute Value

Recall that the distance between two points a and b on a number line is $|a - b|$.

Key Interpretations:

- $|f(x) - L| < \varepsilon$ means: "The distance between $f(x)$ and L is less than ε"
- $0 < |x - a| < \delta$ means: "$x \neq a$ and the distance between x and a is less than δ"

2.2 Important Equivalences

- $|f(x) - L| < \varepsilon$ is equivalent to $L - \varepsilon < f(x) < L + \varepsilon$
- $0 < |x - a| < \delta$ is equivalent to $a - \delta < x < a + \delta$ and $x \neq a$

These equivalences help us move between absolute value notation and interval notation.

3 The Epsilon-Delta Definition

Definition 1 (Epsilon-Delta Definition of a Limit)**.** *Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a. Let L be a real number. Then*

$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

3.1 Breaking Down the Definition

Let's understand each part:

Mathematical Statement	**Translation**
For every $\varepsilon > 0$	For every positive distance ε from L
there exists a $\delta > 0$	there is a positive distance δ from a
such that	such that
if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$	if x is closer than δ to a (and $x \neq a$), then $f(x)$ is closer than ε to L

In plain English: No matter how small a tolerance ε you give me around L, I can find a tolerance δ around a (excluding a itself) such that whenever x is within δ of a, the function value $f(x)$ is guaranteed to be within ε of L.

3.2 The Challenge

The key is that this must work for *every* $\varepsilon > 0$, no matter how small. We must be able to find a corresponding δ that "works."

4 Proving Limits Using Epsilon-Delta

4.1 Problem-Solving Strategy

To prove $\lim_{x \to a} f(x) = L$:

1. Begin with: "Let $\varepsilon > 0$."
2. Find and state: "Choose $\delta =$ _______."
3. Assume: "$0 < |x - a| < \delta$."
4. Show that: "$|f(x) - L| < \varepsilon$."
5. Conclude: "Therefore, $\lim_{x \to a} f(x) = L$."

4.2 Linear Function Example

Example 1 (Proving a Limit for a Linear Function)**.** *Prove that* $\lim_{x \to 1}(2x + 1) = 3$.

Solution:
Step 1: *Let* $\varepsilon > 0$.
Step 2: *Choose* $\delta = \dfrac{\varepsilon}{2}$.
Where does this come from? *We work backwards from what we want:*

$$\begin{aligned} |(2x+1) - 3| &< \varepsilon \\ |2x - 2| &< \varepsilon \\ |2(x-1)| &< \varepsilon \\ 2|x - 1| &< \varepsilon \\ |x - 1| &< \frac{\varepsilon}{2} \end{aligned}$$

So if we make $|x - 1| < \dfrac{\varepsilon}{2}$*, we'll get what we want. Hence,* $\delta = \dfrac{\varepsilon}{2}$.
Step 3: *Assume* $0 < |x - 1| < \delta$.
Step 4: *We need to show* $|(2x + 1) - 3| < \varepsilon$*:*

$$\begin{aligned} |(2x+1) - 3| &= |2x - 2| \\ &= |2(x-1)| \\ &= 2|x - 1| \\ &< 2 \cdot \delta \quad \textit{(using our assumption)} \\ &= 2 \cdot \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Step 5: *Therefore,* $\lim_{x \to 1}(2x + 1) = 3$. □

4.3 Practice Problem

Work this out: Prove that $\lim_{x\to-1}(4x+1) = -3$ by filling in the blanks:

Let $\varepsilon > 0$.

Choose $\delta =$ ________.

Assume $0 < |x - (\_\_\_)| < \delta$.

Thus, $|(4x+1) - (\_\_\_)| = |\_\_\_\_| = \_\_\_|x+1| < \_\_\_\delta = \_\_\_\varepsilon$.

Therefore, $\lim_{x\to-1}(4x+1) = -3$.

Example 2 (Another Linear Function)**.** *Prove that* $\lim_{x\to2}(3x-2) = 4$.

Solution:

Let $\varepsilon > 0$.

Working backwards: We want $|(3x-2)-4| < \varepsilon$*, which gives us* $|3x-6| < \varepsilon$*, so* $3|x-2| < \varepsilon$*, thus* $|x-2| < \frac{\varepsilon}{3}$.

Choose $\delta = \frac{\varepsilon}{3}$.

Assume $0 < |x-2| < \delta$.

Then:

$$\begin{aligned}
|(3x-2)-4| &= |3x-6| \\
&= 3|x-2| \\
&< 3\delta \\
&= 3 \cdot \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

Therefore, $\lim_{x\to2}(3x-2) = 4$. □

5 Quadratic Functions: A More Complex Case

For nonlinear functions, finding δ requires more work.

Example 3 (Quadratic Function)**.** *Prove that* $\lim_{x\to2} x^2 = 4$.

Solution:

Let $\varepsilon > 0$.

Without loss of generality, assume $\varepsilon \leq 4$*. (If our choice works for small* ε*, it works for large* ε *too.)*

Choose $\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\}$.

This choice comes from solving $|x^2 - 4| < \varepsilon$ for x near 2, which gives:

$$4 - \varepsilon < x^2 < 4 + \varepsilon$$

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

The distances from 2 to these bounds give us our δ.
Assume $0 < |x - 2| < \delta$.
Then $|x - 2| < \delta$ implies:

$$-\delta < x - 2 < \delta$$

Since $\delta \leq 2 - \sqrt{4 - \varepsilon}$ and $\delta \leq \sqrt{4 + \varepsilon} - 2$:

$$-(2 - \sqrt{4 - \varepsilon}) \leq -\delta < x - 2 < \delta \leq \sqrt{4 + \varepsilon} - 2$$

Simplifying:

$$-2 + \sqrt{4 - \varepsilon} < x - 2 < \sqrt{4 + \varepsilon} - 2$$

Adding 2:

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

Squaring (all positive):

$$4 - \varepsilon < x^2 < 4 + \varepsilon$$

Subtracting 4:

$$-\varepsilon < x^2 - 4 < \varepsilon$$

Therefore, $|x^2 - 4| < \varepsilon$.
Thus, $\lim_{x \to 2} x^2 = 4$. □

6 Algebraic Approach for Polynomials

For more complex functions, a purely algebraic approach is often simpler.

Example 4 (Algebraic Approach)**.** *Prove that $\lim_{x \to -1}(x^2 - 2x + 3) = 6$.*

Solution:
Let $\varepsilon > 0$.
Choose $\delta = \min\{1, \frac{\varepsilon}{5}\}$.
Why this choice? *We want $|(x^2 - 2x + 3) - 6| < \varepsilon$, which simplifies to $|x^2 - 2x - 3| < \varepsilon$. Factoring: $|x^2 - 2x - 3| = |(x + 1)(x - 3)| = |x + 1| \cdot |x - 3|$.*

If we restrict $|x + 1| < 1$, then $-1 < x + 1 < 1$, so $-2 < x < 0$, which means $-5 < x - 3 < -3$. Thus $|x - 3| < 5$.

Therefore, $|x + 1| \cdot |x - 3| < |x + 1| \cdot 5$. To make this less than ε, we need $|x + 1| < \frac{\varepsilon}{5}$.
Hence, $\delta = \min\{1, \frac{\varepsilon}{5}\}$.
Assume $0 < |x + 1| < \delta$.
Then $|x + 1| < 1$ and $|x + 1| < \frac{\varepsilon}{5}$.
Since $|x + 1| < 1$: $-1 < x + 1 < 1$, so $-5 < x - 3 < -3$, thus $|x - 3| < 5$.

Therefore:

$$
\begin{aligned}
|(x^2 - 2x + 3) - 6| &= |x^2 - 2x - 3| \\
&= |(x+1)(x-3)| \\
&= |x+1| \cdot |x-3| \\
&< \frac{\varepsilon}{5} \cdot 5 \\
&= \varepsilon
\end{aligned}
$$

Therefore, $\lim_{x \to -1}(x^2 - 2x + 3) = 6$. □

6.1 Practice Problem

Work this out: Complete the proof that $\lim_{x \to 1} x^2 = 1$.

Let $\varepsilon > 0$; choose $\delta = \min\{1, \frac{\varepsilon}{3}\}$; assume $0 < |x - 1| < \delta$.

Since $|x - 1| < 1$, we have $-1 < x - 1 < 1$, so $0 < x < 2$, thus $1 < x + 1 < 3$, and $|x + 1| < 3$.

7 One-Sided Limits

The epsilon-delta definition can be modified for one-sided limits.

Definition 2 (Limit from the Right)**.** $\lim_{x \to a^+} f(x) = L$ *if for every* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that if* $0 < x - a < \delta$*, then* $|f(x) - L| < \varepsilon$*.*

Definition 3 (Limit from the Left)**.** $\lim_{x \to a^-} f(x) = L$ *if for every* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that if* $-\delta < x - a < 0$*, then* $|f(x) - L| < \varepsilon$*.*

Note: The only difference is the inequality on $x - a$.

Example 5 (Right-Hand Limit)**.** *Prove that* $\lim_{x \to 4^+} \sqrt{x - 4} = 0$*.*

Solution:
Let $\varepsilon > 0$*.*
Choose $\delta = \varepsilon^2$*.*
We want $|\sqrt{x-4} - 0| < \varepsilon$*, i.e.,* $\sqrt{x-4} < \varepsilon$*, which means* $x - 4 < \varepsilon^2$*, i.e.,* $0 < x - 4 < \varepsilon^2$*.*
Assume $0 < x - 4 < \delta = \varepsilon^2$*.*
Then:

$$|\sqrt{x-4} - 0| = \sqrt{x-4} < \sqrt{\varepsilon^2} = \varepsilon$$

Therefore, $\lim_{x \to 4^+} \sqrt{x - 4} = 0$*.* □

8 Infinite Limits

The epsilon-delta definition can also be adapted for infinite limits.

Definition 4 (Infinite Limit)**.** $\lim_{x\to a} f(x) = +\infty$ *if for every* $M > 0$*, there exists a* $\delta > 0$ *such that if* $0 < |x - a| < \delta$*, then* $f(x) > M$*.*

Interpretation: No matter how large M is, we can make $f(x)$ exceed M by taking x sufficiently close to a.

9 Using Epsilon-Delta to Prove Limit Laws

The epsilon-delta definition is the tool used to prove all the limit laws we've been using.

Theorem 1 (Sum Law)**.** *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$*, then* $\lim_{x\to a}(f(x) + g(x)) = L + M$*.*

Proof. Let $\varepsilon > 0$.

Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$, then $|f(x)-L| < \frac{\varepsilon}{2}$.

Since $\lim_{x\to a} g(x) = M$, there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|g(x)-M| < \frac{\varepsilon}{2}$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.
Assume $0 < |x - a| < \delta$.
Then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$.
Therefore:

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| \quad \text{(Triangle Inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Therefore, $\lim_{x\to a}(f(x) + g(x)) = L + M$. □ □

10 The Triangle Inequality

Definition 5 (Triangle Inequality)**.** *For any real numbers* a *and* b*:* $|a + b| \le |a| + |b|$

This is a crucial tool in epsilon-delta proofs, as we just saw!

11 Summary

- The epsilon-delta definition makes the concept of a limit mathematically precise

- $\lim_{x \to a} f(x) = L$ means: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$
- To prove limits: Let $\varepsilon > 0$, find δ, assume the hypothesis, show the conclusion
- For linear functions, δ is typically ε divided by the coefficient
- For nonlinear functions, we often need $\delta = \min\{\text{bound}, \frac{\varepsilon}{\text{factor}}\}$
- The definition extends naturally to one-sided and infinite limits
- All limit laws can be proven using the epsilon-delta definition