

Derivatives and the Shape of a Graph

Section 4.5

Calculus I - Lecture Notes

March 17, 2026

Motivating Question

How do derivatives tell us the shape of a graph — where it rises, falls, bends up, or bends down?

We know how to find critical points. But a critical point might be a local max, a local min, or neither. Today we develop two tests — one using f' , one using f'' — to classify critical points and describe the shape of any graph.

1 The First Derivative Test

Recall: $f' > 0$ on an interval $\Rightarrow f$ increasing; $f' < 0 \Rightarrow f$ decreasing. Critical points (where $f' = 0$ or is undefined) are the only places f can switch direction, so they divide the line into intervals we test.

Theorem 1 (First Derivative Test). *Suppose f is continuous near a critical point c .*

- (i) *If f' changes from **positive to negative** at c : **local maximum**.*
- (ii) *If f' changes from **negative to positive** at c : **local minimum**.*
- (iii) *If f' has the **same sign** on both sides: **neither**.*

Strategy: Find critical points \rightarrow build sign chart for f' \rightarrow classify by sign changes.

Example 1. *Find all local extrema of $f(x) = x^3 - 3x^2 - 9x - 1$.*

Solution:

$$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$$

Critical points: $x = -1$ and $x = 3$.

<i>Interval</i>	<i>Test Point</i>	<i>Sign of f'</i>	<i>Conclusion</i>
$(-\infty, -1)$	$x = -2$	$3(-)(-) = +$	<i>increasing</i>
$(-1, 3)$	$x = 0$	$3(-)(+) = -$	<i>decreasing</i>
$(3, \infty)$	$x = 4$	$3(+)(+) = +$	<i>increasing</i>

f' changes $+ \rightarrow -$ at $x = -1$: **local max** $f(-1) = -1 - 3 + 9 - 1 = 4$.
 f' changes $- \rightarrow +$ at $x = 3$: **local min** $f(3) = 27 - 27 - 27 - 1 = -28$.

Example 2. Find all local extrema of $f(x) = 5x^{1/3} - x^{5/3}$.

Solution:

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5(1 - x^{4/3})}{3x^{2/3}}$$

$f'(x) = 0$: numerator zero gives $x = \pm 1$. f' undefined at $x = 0$ (critical point since $f(0) = 0$ exists).

The denominator $3x^{2/3} > 0$ always, so sign of f' follows the numerator $5(1 - x^{4/3})$:

<i>Interval</i>	<i>Test Point</i>	<i>Sign of f'</i>	<i>Conclusion</i>
$(-\infty, -1)$	$x = -8$	$5(1 - 16) < 0$	<i>decreasing</i>
$(-1, 0)$	$x = -1/8$	$5(1 - 1/16) > 0$	<i>increasing</i>
$(0, 1)$	$x = 1/8$	$5(1 - 1/16) > 0$	<i>increasing</i>
$(1, \infty)$	$x = 8$	$5(1 - 16) < 0$	<i>decreasing</i>

$x = -1$: $- \rightarrow + \Rightarrow$ **local min**, $f(-1) = -5 + 1 = -4$.

$x = 0$: $+ \rightarrow + \Rightarrow$ **no extremum**.

$x = 1$: $+ \rightarrow - \Rightarrow$ **local max**, $f(1) = 5 - 1 = 4$.

1.1 Practice Problem

Find all local extrema of $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Solution:

$$f'(x) = -3x^2 + 3x + 18 = -3(x - 3)(x + 2)$$

Critical points: $x = -2$, $x = 3$.

<i>Interval</i>	<i>Test Point</i>	<i>Sign of f'</i>	<i>Conclusion</i>
$(-\infty, -2)$	$x = -3$	$-3(-)(-) = -$	decreasing
$(-2, 3)$	$x = 0$	$-3(-)(+) = +$	increasing
$(3, \infty)$	$x = 4$	$-3(+)(+) = -$	decreasing

$x = -2$: $- \rightarrow + \Rightarrow$ **local min**, $f(-2) = 8 + 6 - 36 = -22$.

$x = 3$: $+ \rightarrow - \Rightarrow$ **local max**, $f(3) = -27 + \frac{27}{2} + 54 = \frac{81}{2} = 40.5$.

2 Concavity and Inflection Points

Definition 1 (Concavity). f is **concave up** on I if f' is increasing on I (bowl shape); **concave down** if f' is decreasing (arch shape).

Theorem 2 (Test for Concavity). If $f''(x) > 0$ on I , then f is concave up. If $f''(x) < 0$ on I , then f is concave down.

Definition 2 (Inflection Point). $(a, f(a))$ is an **inflection point** if f is continuous at a and f'' changes sign at a .

Warning: $f''(a) = 0$ is necessary but not sufficient — check that f'' actually changes sign. ($f(x) = x^4$ has $f''(0) = 0$ but no inflection point.)

Example 3. For $f(x) = x^3 - 6x^2 + 9x + 30$, find intervals of concavity and any inflection points.

Solution:

$$f''(x) = 6x - 12 = 0 \implies x = 2$$

Interval	Sign of f''	Conclusion
$(-\infty, 2)$	$6(0) - 12 < 0$	concave down
$(2, \infty)$	$6(3) - 12 > 0$	concave up

f'' changes sign at $x = 2$: inflection point at $(2, f(2)) = (2, 32)$.

2.1 Practice Problem

For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals of concavity and any inflection points.

Solution:

$$f''(x) = -6x + 3 = 0 \implies x = \frac{1}{2}$$

On $(-\infty, \frac{1}{2})$: test $x = 0$, $f''(0) = 3 > 0 \implies$ concave up.

On $(\frac{1}{2}, \infty)$: test $x = 1$, $f''(1) = -3 < 0 \implies$ concave down.

f'' changes sign at $x = \frac{1}{2}$: inflection point at $(\frac{1}{2}, f(\frac{1}{2}))$.

$$f(\frac{1}{2}) = -\frac{1}{8} + \frac{3}{8} + 9 = \frac{37}{4}$$

Answer: Concave up on $(-\infty, \frac{1}{2})$; concave down on $(\frac{1}{2}, \infty)$; inflection point at $(\frac{1}{2}, \frac{37}{4})$.

2.2 What f' and f'' Together Tell Us

Sign of f'	Sign of f''	f is ...	and ...
+	+	Increasing	Concave up
+	-	Increasing	Concave down
-	+	Decreasing	Concave up
-	-	Decreasing	Concave down

3 The Second Derivative Test

Theorem 3 (Second Derivative Test). *Suppose $f'(c) = 0$ and f'' is continuous near c .*

(i) $f''(c) > 0 \Rightarrow$ **local minimum**.

(ii) $f''(c) < 0 \Rightarrow$ **local maximum**.

(iii) $f''(c) = 0 \Rightarrow$ **inconclusive** — use the first derivative test.

Why: If $f'(c) = 0$ and $f''(c) < 0$, then f' is decreasing near c , so $f' > 0$ just left and $f' < 0$ just right — the pattern for a max.

Caution: $f''(c) = 0$ tells us nothing: x^4 has a min, $-x^4$ has a max, x^3 has neither — all with $f''(0) = 0$.

Example 4. Find all local extrema of $f(x) = x^5 - 5x^3$.

Solution:

$$f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0 \implies x = 0, \pm\sqrt{3}$$

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3)$$

c	$f''(c)$	Conclusion
$-\sqrt{3}$	$10(-\sqrt{3})(3) = -30\sqrt{3} < 0$	<i>local max</i>
0	0	<i>inconclusive</i>
$\sqrt{3}$	$10(\sqrt{3})(3) = 30\sqrt{3} > 0$	<i>local min</i>

For $x = 0$, use the first derivative test. On $(-\sqrt{3}, 0)$: $f'(-1) = 5(1)(-2) = -10 < 0$. On $(0, \sqrt{3})$: $f'(1) = -10 < 0$. No sign change \Rightarrow **no extremum at $x = 0$** .

Extremal values: $f(-\sqrt{3}) = -9\sqrt{3} + 15\sqrt{3} = 6\sqrt{3}$ (max); $f(\sqrt{3}) = -6\sqrt{3}$ (min).

3.1 Practice Problem

For $f(x) = x^3 - \frac{3}{2}x^2 - 18x$, classify $x = 3$ and $x = -2$ using the second derivative test.

Solution:

$$f'(x) = 3x^2 - 3x - 18 = 3(x - 3)(x + 2) \quad \checkmark$$

$$f''(x) = 6x - 3$$

$$f''(3) = 15 > 0 \Rightarrow \text{local min.} \quad f(3) = 27 - \frac{27}{2} - 54 = -\frac{81}{2} = -40.5.$$

$$f''(-2) = -15 < 0 \Rightarrow \text{local max.} \quad f(-2) = -8 - 6 + 36 = 22.$$

Section 4.6 (First Half): Limits at Infinity and Horizontal Asymptotes

4 Limits at Infinity

So far our limits have been *local* — we zoomed in near a point $x = a$. Now we ask: what happens to $f(x)$ as x grows without bound in either direction?

Definition 3 (Limit at Infinity). *We write*

$$\lim_{x \rightarrow \infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x sufficiently large and positive. Similarly,

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x sufficiently large and negative.

Intuition: We are describing the *end behavior* of the graph — what happens at the far left and far right.

Definition 4 (Horizontal Asymptote). *The line $y = L$ is a **horizontal asymptote** of f if*

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L.$$

A function can have at most two horizontal asymptotes (one in each direction), and it may cross a horizontal asymptote for finite values of x .

5 End Behavior of Polynomials

For polynomials, end behavior is completely determined by the **leading term**.

Theorem 4 (End Behavior of Polynomials). *Let $p(x) = a_n x^n + \dots$ with $a_n \neq 0$. Then*

$$\lim_{x \rightarrow \pm\infty} p(x) = \lim_{x \rightarrow \pm\infty} a_n x^n.$$

The four cases depending on the sign of a_n and the parity of n :

Leading term	As $x \rightarrow +\infty$	As $x \rightarrow -\infty$	Shape
$a_n > 0, n$ even	$+\infty$	$+\infty$	Both ends up
$a_n < 0, n$ even	$-\infty$	$-\infty$	Both ends down
$a_n > 0, n$ odd	$+\infty$	$-\infty$	Up right, down left
$a_n < 0, n$ odd	$-\infty$	$+\infty$	Down right, up left

Key point: Polynomials never have horizontal asymptotes — they always go to $\pm\infty$.

Example 5. *Describe the end behavior of $p(x) = -2x^3 + 5x^2 - 1$.*

Leading term $-2x^3$: $\lim_{x \rightarrow +\infty} p(x) = -\infty$ and $\lim_{x \rightarrow -\infty} p(x) = +\infty$. Falls right, rises left. No horizontal asymptote.

6 Limits at Infinity for Rational Functions

Rational functions $f(x) = p(x)/q(x)$ behave differently depending on the degrees of numerator and denominator.

The key technique: Divide numerator and denominator by the highest power of x appearing in the denominator, then use the fact that

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^k} = 0 \quad \text{for any } k > 0.$$

Theorem 5 (Horizontal Asymptotes of Rational Functions). Let $f(x) = \frac{a_n x^n + \dots}{b_m x^m + \dots}$.

(i) If $n < m$ (degree of numerator is less): $\lim_{x \rightarrow \pm\infty} f(x) = 0$. Horizontal asymptote $y = 0$.

(ii) If $n = m$ (same degree): $\lim_{x \rightarrow \pm\infty} f(x) = \frac{a_n}{b_m}$. Horizontal asymptote $y = \frac{a_n}{b_m}$.

(iii) If $n > m$ (degree of numerator is greater): $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$. No horizontal asymptote.

Example 6. Find all horizontal asymptotes of $f(x) = \frac{3x^2 - 5}{2x^2 + 1}$.

Both numerator and denominator have degree 2. Divide by x^2 :

$$\lim_{x \rightarrow \pm\infty} \frac{3x^2 - 5}{2x^2 + 1} = \lim_{x \rightarrow \pm\infty} \frac{3 - \frac{5}{x^2}}{2 + \frac{1}{x^2}} = \frac{3}{2}$$

Horizontal asymptote: $y = \frac{3}{2}$.

Example 7. Find horizontal asymptotes of $f(x) = \frac{4x - 1}{x^2 + 3}$ and $g(x) = \frac{x^2 - 1}{3x + 2}$.

For f : degree of numerator (1) < degree of denominator (2), so $\lim_{x \rightarrow \pm\infty} f(x) = 0$. Horizontal asymptote $y = 0$.

For g : degree of numerator (2) > degree of denominator (1), so $\lim_{x \rightarrow \pm\infty} g(x) = \pm\infty$. No horizontal asymptote. (This case leads to an oblique asymptote — covered next class.)

6.1 Practice Problem

Find all horizontal asymptotes of $f(x) = \frac{5x^3 - 2x}{5x^3 + 4x^2 + 1}$.

Solution: Both have degree 3. Divide by x^3 :

$$\lim_{x \rightarrow \pm\infty} \frac{5x^3 - 2x}{5x^3 + 4x^2 + 1} = \lim_{x \rightarrow \pm\infty} \frac{5 - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^3}} = \frac{5}{5} = 1$$

Horizontal asymptote: $y = 1$.

7 Summary

First Derivative Test: To classify a critical point c , check the sign of f' on both sides.

f' changes $+$ \rightarrow $-$ at c	Local maximum
f' changes $-$ \rightarrow $+$ at c	Local minimum
f' same sign on both sides	Neither

Concavity Test:

- $f''(x) > 0$ on $I \Rightarrow f$ is concave up on I
- $f''(x) < 0$ on $I \Rightarrow f$ is concave down on I
- Inflection point where f'' changes sign (and f is continuous)

Second Derivative Test: If $f'(c) = 0$:

- $f''(c) > 0 \Rightarrow$ local **minimum**
- $f''(c) < 0 \Rightarrow$ local **maximum**
- $f''(c) = 0 \Rightarrow$ **inconclusive** (fall back to the first derivative test)

Limits at Infinity / Horizontal Asymptotes:

- Polynomials: end behavior determined by leading term; no horizontal asymptotes
- Rational functions: divide by highest power in denominator; $y = a_n/b_m$ when degrees are equal, $y = 0$ when numerator degree is smaller, no HA when numerator degree is larger

Next time (March 19): Section 4.7 — Applied Optimization. We will use everything from 4.5–4.6 to solve real-world maximum and minimum problems.