Limits at Infinity & Applied Optimization
Sections 4.6–4.7
Calculus I - Lecture Notes

March 19, 2026

Motivating Question

Spotify has 600 million users. How should they price a subscription to maximize revenue?

Suppose market research tells Spotify that if they charge p dollars per month, the number of paying subscribers (in millions) is approximately $n(p) = 200 - 8p$. Revenue is $R = n \cdot p$. What price p maximizes R?

This is an **optimization problem** — one of the central applications of calculus. Before we get there, we finish the toolkit from Section 4.6: oblique asymptotes and the end behavior of transcendental functions. Then we build a complete problem-solving strategy and work through several applied examples.

Section 4.6 (Second Half): Oblique Asymptotes and Transcendental Functions

1 Oblique Asymptotes

Recall from last class: when the degree of the numerator exceeds the degree of the denominator in a rational function, there is no horizontal asymptote. Instead, the graph approaches a **slant (oblique) asymptote** — a line $y = mx + b$ with $m \neq 0$.

How to find it: Perform polynomial long division of numerator by denominator. The quotient (ignoring the remainder) is the oblique asymptote.

Example 1. *Find all asymptotes of* $f(x) = \dfrac{x^2 - 1}{x - 2}$.

Solution:

The numerator has degree 2 and denominator degree 1, so there is no horizontal asymptote.

Perform polynomial long division:

$$x^2 - 1 \div (x-2):$$

$$x^2 - 1 = (x-2)(x+2) + 3 \implies f(x) = x + 2 + \frac{3}{x-2}.$$

As $x \to \pm\infty$, the remainder $\frac{3}{x-2} \to 0$, so $f(x) \to x+2$.

Oblique asymptote: $y = x + 2$.

Vertical asymptote: $x = 2$ *(denominator is zero, numerator is $3 \neq 0$ there).*

1.1 Practice Problem

Find the oblique asymptote of $f(x) = \frac{2x^2+3x-1}{x+1}$.

Solution:

Divide $2x^2 + 3x - 1$ by $x + 1$:

$$2x^2 + 3x - 1 = (x+1)(2x+1) + (-2) \implies f(x) = 2x + 1 - \frac{2}{x+1}.$$

As $x \to \pm\infty$, the remainder vanishes.

Oblique asymptote: $y = 2x + 1$. **Vertical asymptote:** $x = -1$.

Note: Transcendental Functions (Section 4.6, not covered)

Section 4.6 in the textbook also discusses limits at infinity for transcendental functions — exponentials, logarithms, and their combinations with polynomials (e.g., e^x, $\ln x$, x^n/e^x). **We do not cover this material in Math 1231.** If you are curious, the relevant examples are in Section 4.6 of the OpenStax text.

Section 4.7: Applied Optimization

Back to Spotify

We posed the revenue question: $n(p) = 200 - 8p$ (millions of subscribers), so

$$R(p) = p \cdot n(p) = p(200 - 8p) = 200p - 8p^2.$$

We need $n \geq 0$, so $p \leq 25$. Also $p \geq 0$. Domain: $[0, 25]$.

$$R'(p) = 200 - 16p = 0 \implies p = 12.50.$$

$R''(p) = -16 < 0$, confirming a maximum. The optimal price is \$12.50/month, generating $R(12.50) = 200(12.50) - 8(156.25) = 2500 - 1250 = \$1250$ million per month.

This is the core idea of optimization: **write the objective as a function of one variable, find critical points, verify.**

2 The Optimization Strategy

1. **Introduce variables.** Draw a picture if helpful. Label everything.
2. **Identify the objective.** Which quantity are you maximizing or minimizing?
3. **Write the objective function.** Express it in terms of your variables.
4. **Use the constraint** to eliminate variables until you have a function of *one* variable.
5. **Determine the domain.** What values are physically meaningful?
6. **Optimize.** Find critical points; check critical points and endpoints; state the answer.

When is a critical point a max/min?

- On a **closed interval**: use the Extreme Value Theorem — check all critical points and both endpoints.
- On an **open or unbounded interval**: if $S(x) \to \infty$ at both ends and there is exactly one critical point, that critical point must be the minimum (and vice versa for max).

3 Examples

Example 2 (Maximizing Area — Fencing Problem)**.** *A farmer has 100 ft of fencing to enclose a rectangular garden using a rock wall as one side (so fencing is needed for only three sides). What dimensions maximize the area?*

Solution:

Let x = length of each side perpendicular to the wall, y = length of the side parallel to the wall.

Objective: $A = xy$.

Constraint: $2x + y = 100$, *so* $y = 100 - 2x$.

Reduced objective:

$$A(x) = x(100 - 2x) = 100x - 2x^2.$$

Domain: $x > 0$ *and* $y = 100 - 2x > 0 \Rightarrow x < 50$. *Consider* $[0, 50]$.

Optimize:

$$A'(x) = 100 - 4x = 0 \implies x = 25.$$

$A(0) = 0,\ A(50) = 0,\ A(25) = 100(25) - 2(625) = 2500 - 1250 = 1250.$

Maximum area is 1250 ft^2 *when* $x = 25$ *ft and* $y = 50$ *ft.*

Example 3 (Maximizing Volume — Open Box). *A 24×36 in. piece of cardboard has squares of side x cut from each corner; the flaps are folded up to form an open-top box. What value of x maximizes the volume?*

Solution:

After removing the corners: height $= x$, length $= 36 - 2x$, width $= 24 - 2x$.

Objective:

$$V(x) = x(36 - 2x)(24 - 2x) = 4x^3 - 120x^2 + 864x.$$

Domain: *$x > 0$ and $24 - 2x > 0 \Rightarrow x < 12$. Consider $[0, 12]$.*

Optimize:

$$V'(x) = 12x^2 - 240x + 864 = 12(x^2 - 20x + 72).$$

Quadratic formula:

$$x = \frac{20 \pm \sqrt{400 - 288}}{2} = \frac{20 \pm \sqrt{112}}{2} = 10 \pm 2\sqrt{7}.$$

$10 + 2\sqrt{7} \approx 15.3$ is outside $[0, 12]$. The only relevant critical point is $x = 10 - 2\sqrt{7} \approx 4.7$ in.

Since $V(0) = V(12) = 0$ and $V > 0$ on $(0, 12)$, the maximum occurs at the interior critical point.

Maximum volume: *$V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825$ in.^{3}*

Example 4 (Minimizing Surface Area — Unbounded Domain). *A rectangular box with a square base, open top, and volume 216 in.^{3} is to be built. What dimensions minimize surface area?*

Solution:

Let x = side length of square base, y = height.

Objective: *$S = x^2 + 4xy$ (base + four sides).*

Constraint: *$x^2 y = 216$, so $y = \dfrac{216}{x^2}$.*

Reduced objective:

$$S(x) = x^2 + 4x \cdot \frac{216}{x^2} = x^2 + \frac{864}{x}.$$

Domain: $(0, \infty)$.

Optimize:

$$S'(x) = 2x - \frac{864}{x^2} = 0 \implies 2x^3 = 864 \implies x^3 = 432 \implies x = 6\sqrt[3]{2}.$$

Verify it's a minimum: *As $x \to 0^+$, $S \to \infty$; as $x \to \infty$, $S \to \infty$. With exactly one critical point on $(0, \infty)$, it must be the global minimum.*

Dimensions: *$x = 6\sqrt[3]{2}$ in., $y = \dfrac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$ in.*

Minimum surface area: *$S = 108\sqrt[3]{4}$ in.^{2}*

Note: *The height is exactly half the base side length. This ratio $y = x/2$ holds for any open-top square box minimizing surface area for a given volume.*

Example 5 (Maximizing Revenue)**.** *A car rental company charges p dollars/day,* $50 \le p \le 200$*. Number of cars rented per day:* $n(p) = 1000 - 5p$*. What price maximizes revenue?*

Solution:

$R(p) = p \cdot n(p) = p(1000 - 5p) = 1000p - 5p^2$.

$R'(p) = 1000 - 10p = 0 \implies p = 100$.

Check endpoints: $R(50) = 37{,}500$, $R(100) = 50{,}000$, $R(200) = 0$.

Charge $100/day *to maximize revenue at $50,000/day.*

3.1 Practice Problem

A farmer wants to build a rectangular pen using 400 ft of fencing, with a river forming one side (no fencing needed there). What dimensions maximize the enclosed area?

Solution:

Let x = sides perpendicular to the river, y = side parallel to the river (no fence needed).

Constraint: $2x + y = 400$, so $y = 400 - 2x$.

Objective:

$$A(x) = xy = x(400 - 2x) = 400x - 2x^2.$$

Domain: $0 < x < 200$.

$$A'(x) = 400 - 4x = 0 \implies x = 100.$$

$A(0) = A(200) = 0$; $A(100) = 400(100) - 2(10{,}000) = 40{,}000 - 20{,}000 = 20{,}000$.

Answer: $x = 100$ ft, $y = 200$ ft. Maximum area is **20,000 ft**^{2}.

4 Summary

Second half of 4.6:

- **Oblique asymptote** when deg(numerator) = deg(denominator) + 1: divide to find $y = mx + b$.
- End behavior of transcendental functions (exponentials, logs) is in the textbook but not covered in Math 1231.

Optimization strategy (4.7):

1. Name variables, draw picture.
2. Write objective function.
3. Use constraint to reduce to one variable.
4. Determine domain.
5. Find and classify critical points; check endpoints if domain is closed.

6. If domain is $(0, \infty)$ and $f \to \infty$ at both ends: unique critical point is the global min (or max by symmetry).