Approximating Areas
Section 5.1
Calculus I - Lecture Notes

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Motivating Question

How do we find the area under a curve when we can't use simple geometry?

We know how to find areas of rectangles, triangles, and circles. But what about the area bounded by $y = x^2$ above, the x-axis below, and the lines $x = 0$ and $x = 2$? There's no simple formula. Our strategy: approximate using shapes we *do* understand, then take a limit.

Motivating Example: How Far Does a Car Travel?

Suppose a car accelerates from rest so that its speed at time t seconds is $v(t) = t^2$ feet per second. How far does it travel in the first 2 seconds?

If the speed were *constant* at v ft/s, distance would just be $v \times t$—a rectangle. But $v(t) = t^2$ is changing. What we can do is break the 2 seconds into small intervals and pretend the speed is constant on each one:

Interval	Speed at left endpoint	Width	Contribution
$[0, 0.5]$	$0^2 = 0$ ft/s	0.5 s	0 ft
$[0.5, 1]$	$0.5^2 = 0.25$ ft/s	0.5 s	0.125 ft
$[1, 1.5]$	$1^2 = 1$ ft/s	0.5 s	0.5 ft
$[1.5, 2]$	$1.5^2 = 2.25$ ft/s	0.5 s	1.125 ft
		Total estimate:	1.75 ft

This is an underestimate (we used the slower speed at the start of each interval). With more intervals, the estimate improves. In the limit, distance $= \int_0^2 t^2\,dt$—which is exactly the area under $y = t^2$ from 0 to 2. This is the connection we are building today.

1 Sigma Notation

Before we can discuss area approximations, we need compact notation for sums.

1.1 The Basic Idea

Definition 1. *The sum* $a_1 + a_2 + a_3 + \cdots + a_n$ *is written in* ***sigma notation*** *as*

$$\sum_{i=1}^{n} a_i$$

where i *is the* ***index*** *(a dummy variable),* 1 *is the lower limit, and* n *is the upper limit.*

Example 1. *Write out the sum* $\sum_{i=1}^{5} 3^i$ *and evaluate it.*

Solution:

$$\begin{aligned}\sum_{i=1}^{5} 3^i &= 3^1 + 3^2 + 3^3 + 3^4 + 3^5 \\ &= 3 + 9 + 27 + 81 + 243 = 363\end{aligned}$$

Example 2. *Write* $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$ *in sigma notation.*

Solution:
Each term has the form $\frac{1}{i^2}$ *for* $i = 1, 2, 3, 4, 5$*:*

$$\sum_{i=1}^{5} \frac{1}{i^2}$$

1.2 Properties of Sigma Notation

Theorem 1 (Properties of Sigma Notation)**.** *Let* $\{a_i\}$ *and* $\{b_i\}$ *be sequences and* c *a constant. For all positive integers* n*:*

1. $$\sum_{i=1}^{n} c = nc$$
2. $$\sum_{i=1}^{n} c\,a_i = c\sum_{i=1}^{n} a_i$$
3. $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$
4. $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

1.3 Closed-Form Summation Formulas

Theorem 2 (Sums and Powers of Integers)**.**

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

These formulas let us evaluate sums involving large n without adding up every term.

Example 3. *Evaluate* $\sum_{i=1}^{200}(i-3)^2$.

Solution:
Expand $(i-3)^2 = i^2 - 6i + 9$ *and apply linearity:*

$$\begin{aligned}\sum_{i=1}^{200}(i-3)^2 &= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9\\ &= \frac{200\cdot 201\cdot 401}{6} - 6\cdot\frac{200\cdot 201}{2} + 9\cdot 200\\ &= 2{,}686{,}700 - 120{,}600 + 1{,}800\\ &= 2{,}567{,}900\end{aligned}$$

1.4 Practice Problem

Work this out: Evaluate $\sum_{k=1}^{20}(2k+1)$.

Solution:
Split using linearity and apply the closed-form formulas:

$$\begin{aligned}\sum_{k=1}^{20}(2k+1) &= 2\sum_{k=1}^{20} k + \sum_{k=1}^{20} 1\\ &= 2\cdot\frac{20\cdot 21}{2} + 20\\ &= 420 + 20 = \mathbf{440}\end{aligned}$$

2 Approximating Area with Rectangles

2.1 Setting Up a Partition

Let $f(x)$ be continuous and nonnegative on $[a, b]$. We want to approximate the area A between the curve and the x-axis.

Definition 2. *A* ***regular partition*** *of $[a, b]$ into n subintervals divides the interval at equally spaced points*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

with common width $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$.

2.2 Left-Endpoint and Right-Endpoint Approximations

Theorem 3 (Left-Endpoint Approximation)**.** *On each subinterval $[x_{i-1}, x_i]$, build a rectangle of height $f(x_{i-1})$ and width Δx. The* ***left-endpoint approximation*** *with n rectangles is*

$$A \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

Theorem 4 (Right-Endpoint Approximation)**.** *On each subinterval $[x_{i-1}, x_i]$, build a rectangle of height $f(x_i)$ and width Δx. The* ***right-endpoint approximation*** *with n rectangles is*

$$A \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Example 4. *Approximate the area under $f(x) = x^2$ on $[0, 2]$ using $n = 4$ subintervals with both left- and right-endpoint methods.*

Solution:
We have $\Delta x = \dfrac{2-0}{4} = 0.5$ and partition points $x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2$.

Left-endpoint *(use x_0, x_1, x_2, x_3):*

$$\begin{aligned} L_4 &= f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 0 + 0.125 + 0.5 + 1.125 = 1.75 \end{aligned}$$

Right-endpoint *(use x_1, x_2, x_3, x_4):*

$$\begin{aligned} R_4 &= f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 0.125 + 0.5 + 1.125 + 2 = 3.75 \end{aligned}$$

Since $f(x) = x^2$ is increasing, the left sum underestimates *and the right sum* overestimates*. The true area lies between* 1.75 *and* 3.75*.*

What happens as n increases? More rectangles means thinner strips and better approximations:

n	L_n	R_n
4	1.75	3.75
8	2.1875	3.1875
32	2.5508	2.9258

Both sequences appear to converge toward $\frac{8}{3} \approx 2.667$. We'll confirm this exactly when we define the definite integral.

2.3 Practice Problem

Work this out: Approximate the area under $f(x) = \frac{1}{x}$ on $[1, 2]$ using $n = 4$ left-endpoint rectangles.

Solution:
We have $\Delta x = \frac{2-1}{4} = 0.25$ and left endpoints $x_0 = 1,\ x_1 = 1.25,\ x_2 = 1.5,\ x_3 = 1.75$.

$$\begin{aligned} L_4 &= f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) \\ &= \frac{1}{1}(0.25) + \frac{1}{1.25}(0.25) + \frac{1}{1.5}(0.25) + \frac{1}{1.75}(0.25) \\ &= 0.25\left(1 + 0.8 + 0.\overline{6} + 0.\overline{571428}\right) \\ &\approx 0.25(1 + 0.8 + 0.6667 + 0.5714) \\ &\approx 0.25 \times 3.0381 \approx \mathbf{0.7595} \end{aligned}$$

Since $f(x) = 1/x$ is decreasing, the left-endpoint sum *overestimates* the true area. (The exact answer is $\ln 2 \approx 0.6931$, which we'll be able to confirm once we have the Fundamental Theorem.)

3 Riemann Sums

The left- and right-endpoint methods require evaluating f at specific points. But we can use *any* point in each subinterval.

Definition 3. *Let $f(x)$ be defined on $[a, b]$ and let P be a regular partition. For each subinterval $[x_{i-1}, x_i]$, choose any sample point x_i^*. A **Riemann sum** for f on $[a, b]$ is*

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Special cases:

- $x_i^* = x_{i-1}$ (left endpoint) gives the left-endpoint approximation L_n

- $x_i^* = x_i$ (right endpoint) gives the right-endpoint approximation R_n
- $x_i^* = \frac{x_{i-1}+x_i}{2}$ (midpoint) gives the midpoint approximation M_n

3.1 Upper and Lower Sums

Definition 4. *An* ***upper sum*** *chooses x_i^* to maximize f on each subinterval (giving the maximum rectangle height). A* ***lower sum*** *chooses x_i^* to minimize f. These give an overestimate and underestimate respectively.*

Key fact: If f is monotone increasing on $[a,b]$, then L_n is the lower sum and R_n is the upper sum. If f is monotone decreasing, they swap.

Example 5. *Find a lower sum for $f(x) = 10 - x^2$ on $[1,2]$ with $n = 4$.*

Solution:
Here $\Delta x = \dfrac{1}{4}$ and the partition points are $1,\ 1.25,\ 1.5,\ 1.75,\ 2$.
Since $f(x) = 10 - x^2$ is decreasing *on $[1,2]$, the minimum on each subinterval occurs at the* ***right*** *endpoint.*

$$\begin{aligned}
\textit{Lower sum} &= \sum_{k=1}^{4}\big(10 - x_k^2\big)(0.25) \\
&= 0.25\big[(10 - 1.25^2) + (10 - 1.5^2) + (10 - 1.75^2) + (10 - 2^2)\big] \\
&= 0.25\big[8.4375 + 7.75 + 6.9375 + 6\big] \\
&= 0.25 \times 29.125 = 7.28
\end{aligned}$$

This is an underestimate: the true area is at least 7.28.

3.2 Practice Problem

Work this out: For the same function $f(x) = 10 - x^2$ on $[1,2]$ with $n = 4$, find an upper sum.

Solution:
Since $f(x) = 10 - x^2$ is decreasing on $[1,2]$, the maximum on each subinterval occurs at the **left** endpoint. Use left endpoints $x_0 = 1,\ x_1 = 1.25,\ x_2 = 1.5,\ x_3 = 1.75$:

$$\begin{aligned}
\text{Upper sum} &= 0.25\big[(10 - 1^2) + (10 - 1.25^2) + (10 - 1.5^2) + (10 - 1.75^2)\big] \\
&= 0.25\big[9 + 8.4375 + 7.75 + 6.9375\big] \\
&= 0.25 \times 32.125 = \mathbf{8.03125}
\end{aligned}$$

So the true area lies between 7.28 and 8.03125.

4 Exact Area as a Limit

Definition 5. *Let $f(x)$ be continuous and nonnegative on $[a, b]$. The **area under the curve** $y = f(x)$ on $[a, b]$ is*

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Remarkable fact: If f is continuous, this limit exists and equals the same value regardless of how we choose the sample points x_i^*.

Example 6. *Use a right-endpoint Riemann sum to find the exact area under $f(x) = x^2$ on $[0, 2]$.*

Solution:
With $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$:

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n} \\ &= \sum_{i=1}^{n} \frac{4i^2}{n^2} \cdot \frac{2}{n} = \frac{8}{n^3} \sum_{i=1}^{n} i^2 \\ &= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{8(n+1)(2n+1)}{6n^2} \end{aligned}$$

Taking the limit:

$$\begin{aligned} A &= \lim_{n\to\infty} \frac{8(n+1)(2n+1)}{6n^2} \\ &= \lim_{n\to\infty} \frac{8(2n^2+3n+1)}{6n^2} \\ &= \frac{8\cdot 2}{6} = \frac{16}{6} = \boxed{\frac{8}{3}} \end{aligned}$$

This confirms the table from earlier: both L_n and R_n converge to $\frac{8}{3}$.

5 Summary

Sigma notation key formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Area approximation:

- Divide $[a, b]$ into n equal subintervals of width $\Delta x = \frac{b-a}{n}$

- Left: $L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$ Right: $R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$
- For increasing f: $L_n < A < R_n$ (for decreasing: reversed)
- More rectangles ⇒ better approximation

Exact area:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Next class (5.2) we will give this limit a name—the **definite integral**—and develop tools to compute it efficiently.