The Definite Integral
Section 5.2
Calculus I - Lecture Notes

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Motivating Question

Can we extend the area idea to functions that dip below the x-axis, and compute exact values systematically?

Last class we defined area under a nonnegative curve as a limit of Riemann sums. Today we lift the nonnegativity restriction, give the limit a formal name—the **definite integral**—and develop its key properties.

Motivating Example: Profit and Loss

Suppose a company's daily profit rate (in thousands of dollars per day) is modeled by $p(t) = t - 2$ over a 6-day period, $t \in [0, 6]$. The function is negative for the first two days (the company is losing money) and positive after that.

The *net* profit over the 6 days is not just the total area under the curve—it is the area above the axis *minus* the area below. This is what $\int_0^6 (t-2)\,dt$ computes: signed area, or net change.

If instead we want total dollars that *changed hands* (losses counted positively too), we compute $\int_0^6 |t-2|\,dt$. We will make both ideas precise today and see that the distinction between net signed area and total area matters whenever a function changes sign.

1 The Definite Integral: Definition and Notation

Definition 1. *Let $f(x)$ be defined on $[a, b]$. The* ***definite integral of*** f ***from*** a ***to*** b *is*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided this limit exists. When the limit exists, we say f is ***integrable*** *on $[a, b]$.*

Anatomy of the notation:

- $\int$ is an elongated S (for "sum"), introduced by Leibniz
- a and b are the **limits of integration**: a is the lower limit, b is the upper limit
- $f(x)$ is the **integrand**
- dx indicates the variable of integration (a dummy variable—we could write dt or du for the same value)

Critical distinction: A definite integral $\int_a^b f(x)\,dx$ is a *number*. An indefinite integral $\int f(x)\,dx$ is a *family of functions*. Don't confuse them.

Theorem 1 (Continuous Functions Are Integrable)**.** *If $f(x)$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.*

Functions with finitely many jump discontinuities are also integrable, but we won't need that level of generality in this course.

2 Evaluating Definite Integrals from the Definition

Example 1. *Use the definition to evaluate $\int_0^2 x^2\,dx$.*

Solution:
Use right endpoints: $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$, so $f(x_i) = \frac{4i^2}{n^2}$.
Form the Riemann sum:

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \frac{4i^2}{n^2} \cdot \frac{2}{n} = \frac{8}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

Take the limit:

$$\int_0^2 x^2\,dx = \lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3}$$

2.1 Practice Problem

Work this out: Use right-endpoint Riemann sums to evaluate $\int_0^3 (2x-1)\,dx$.

Solution:
Set up the right-endpoint sum with $a=0$, $b=3$: $\Delta x = \frac{3}{n}$ and $x_i = \frac{3i}{n}$.

Compute $f(x_i) = 2x_i - 1 = \dfrac{6i}{n} - 1$.
Form the Riemann sum:

$$R_n = \sum_{i=1}^{n} \left(\frac{6i}{n} - 1\right)\frac{3}{n} = \frac{3}{n}\sum_{i=1}^{n}\frac{6i}{n} - \frac{3}{n}\sum_{i=1}^{n} 1$$

$$= \frac{18}{n^2}\sum_{i=1}^{n} i - \frac{3}{n}\cdot n = \frac{18}{n^2}\cdot\frac{n(n+1)}{2} - 3$$

$$= \frac{9(n+1)}{n} - 3 = 9 + \frac{9}{n} - 3 = 6 + \frac{9}{n}$$

Take the limit:

$$\int_0^3 (2x-1)\,dx = \lim_{n\to\infty}\left(6 + \frac{9}{n}\right) = \boxed{6}$$

3 Evaluating Integrals Using Geometry

Because the definite integral equals area (signed—more on this below), we can often evaluate integrals directly using area formulas for familiar shapes.

Example 2. *Evaluate* $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.

Solution:

The integrand $f(x) = \sqrt{9-(x-3)^2}$ *is the upper half of the circle* $(x-3)^2 + y^2 = 9$ *(radius 3, centered at* $(3,0)$*). Over* $[3,6]$ *this traces the right quarter of that semicircle.*

Area of the full circle: $\pi r^2 = 9\pi$*. One quarter of that:*

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4} \approx 7.069$$

Example 3. *Evaluate* $\int_2^4 (2x+3)\,dx$ *using geometry.*

Solution:

The function $f(x) = 2x+3$ *is a line. The region under it from* $x = 2$ *to* $x = 4$ *is a trapezoid with:*

$$f(2) = 7, \quad f(4) = 11, \quad \textit{width} = 2$$

Using the trapezoid area formula $A = \frac{1}{2}h(a+b)$*:*

$$\int_2^4 (2x+3)\,dx = \frac{1}{2}(2)(7+11) = 18$$

3.1 Practice Problem

Work this out: Evaluate $\int_{-2}^{2} \sqrt{4-x^2}\,dx$ using a geometric formula.

Solution:

The integrand $f(x) = \sqrt{4-x^2}$ is the upper half of the circle $x^2 + y^2 = 4$, which has radius 2. Over $[-2, 2]$ this traces the entire upper semicircle.

Area of a full circle of radius 2: $\pi r^2 = 4\pi$. The upper semicircle is half of that:

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2}\pi(2)^2 = \boxed{2\pi}$$

4 Net Signed Area

When $f(x)$ dips below the x-axis, products $f(x_i^*)\,\Delta x$ become *negative*. The definite integral then measures **net signed area**: area above the axis minus area below.

Definition 2. *Let A_1 be the area between $f(x)$ and the x-axis lying* above *the axis, and A_2 be the area lying* below*. Then:*

$$\textit{Net signed area} = \int_a^b f(x)\,dx = A_1 - A_2$$

$$\textit{Total area} = \int_a^b |f(x)|\,dx = A_1 + A_2$$

Example 4. *Find the net signed area between $f(x) = 2x$ and the x-axis on $[-3, 3]$.*

Solution:

The line $y = 2x$ crosses zero at $x = 0$. On $[-3, 0]$ it is negative (below axis); on $[0, 3]$ positive (above axis). Each region is a triangle.

Area above axis *(A_1, right triangle with base 3 and height $2(3) = 6$):*

$$A_1 = \frac{1}{2}(3)(6) = 9$$

Area below axis *(A_2, same dimensions):*

$$A_2 = \frac{1}{2}(3)(6) = 9$$

Net signed area:

$$\int_{-3}^{3} 2x\,dx = A_1 - A_2 = 9 - 9 = 0$$

Example 5. *Find the total area between $f(x) = x - 2$ and the x-axis on $[0, 6]$.*

Solution:

The x-intercept is at $x = 2$. On $[0, 2]$, $f < 0$; on $[2, 6]$, $f > 0$.

Area below axis *(A_2, triangle with base 2, height 2):*

$$A_2 = \frac{1}{2}(2)(2) = 2$$

Area above axis *(A_1, triangle with base 4, height 4):*

$$A_1 = \frac{1}{2}(4)(4) = 8$$

Net signed area: $A_1 - A_2 = 8 - 2 = 6$
Total area: $A_1 + A_2 = 8 + 2 = 10$

4.1 Practice Problem

Work this out: Find the net signed area and the total area between $f(x) = 2x$ and the x-axis on $[-3, 3]$.

Hint: We found the net signed area above—what is the total area?

Solution:

We already found the net signed area: $\int_{-3}^{3} 2x\, dx = 0$.

For the total area, both triangles (A_1 above and A_2 below) have area 9, so:

$$\int_{-3}^{3} |2x|\, dx = A_1 + A_2 = 9 + 9 = \boxed{18}$$

The net is zero because the negative and positive regions cancel perfectly, but the total area is 18.

5 Properties of the Definite Integral

Theorem 2 (Properties of the Definite Integral)**.**

1. $\int_a^a f(x)\, dx = 0$

2. $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$

3. $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

4. $\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

5. $\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx$

6. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

Properties 1–5 are natural analogues of summation properties. Property 6 is especially useful: it says we can split a single integral over $[a, b]$ at any interior point c.

Example 6. *Express* $\int_{-2}^{1} (-3x^3 + 2x + 2)\, dx$ *as a sum of three integrals.*

Solution:
Apply properties 3 and 5:

$$\begin{aligned}\int_{-2}^{1} (-3x^3 + 2x + 2)\, dx &= \int_{-2}^{1} -3x^3\, dx + \int_{-2}^{1} 2x\, dx + \int_{-2}^{1} 2\, dx \\ &= -3\int_{-2}^{1} x^3\, dx + 2\int_{-2}^{1} x\, dx + \int_{-2}^{1} 2\, dx\end{aligned}$$

Example 7. *Suppose* $\int_{0}^{8} f(x)\, dx = 10$ *and* $\int_{0}^{5} f(x)\, dx = 5$. *Find* $\int_{5}^{8} f(x)\, dx$.

Solution:
By property 6:

$$\int_{0}^{8} f(x)\, dx = \int_{0}^{5} f(x)\, dx + \int_{5}^{8} f(x)\, dx$$

$$10 = 5 + \int_{5}^{8} f(x)\, dx \implies \int_{5}^{8} f(x)\, dx = 5$$

5.1 Practice Problem

Work this out: Given $\int_{1}^{5} f(x)\, dx = -3$ and $\int_{2}^{5} f(x)\, dx = 4$, find $\int_{1}^{2} f(x)\, dx$.

Solution:
By property 6 with $a = 1$, $c = 2$, $b = 5$:

$$\int_{1}^{5} f(x)\, dx = \int_{1}^{2} f(x)\, dx + \int_{2}^{5} f(x)\, dx$$

$$-3 = \int_{1}^{2} f(x)\, dx + 4 \implies \int_{1}^{2} f(x)\, dx = \boxed{-7}$$

6 Comparison Properties

Sometimes we just need to know whether one integral is larger than another, without computing exact values.

Theorem 3 (Comparison Theorem)**.** *Suppose* $a \leq b$.

i. If $f(x) \geq 0$ *on* $[a, b]$, *then* $\int_{a}^{b} f(x)\, dx \geq 0$.

ii. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

iii. If $m \leq f(x) \leq M$ on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Part (iii) is particularly useful: if you can bound f between two constants, you can bound the integral between two rectangles.

7 Average Value of a Function

The definite integral also lets us generalize the notion of average to continuous functions.

Definition 3. *Let $f(x)$ be continuous on $[a, b]$. The **average value** of f on $[a, b]$ is*

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Where does this come from? The average of n sampled values $f(x_1^*), \ldots, f(x_n^*)$ is

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i^*) = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Taking $n \to \infty$ gives the formula above.

Example 8. *Find the average value of $f(x) = x + 1$ on $[0, 5]$.*

Solution:

The region under $y = x + 1$ from 0 to 5 is a trapezoid with parallel sides $f(0) = 1$ and $f(5) = 6$, and width 5.

$$\int_0^5 (x+1)\,dx = \frac{1}{2}(5)(1+6) = \frac{35}{2}$$

Average value:

$$f_{ave} = \frac{1}{5-0} \cdot \frac{35}{2} = \frac{35}{10} = \frac{7}{2}$$

Interpretation: *A horizontal line at height $\frac{7}{2}$ over $[0, 5]$ encloses the same area as $y = x + 1$ does.*

7.1 Practice Problem

Work this out: Find the average value of $f(x) = 6 - 2x$ on $[0, 3]$.

Solution:

The region under $y = 6 - 2x$ from 0 to 3 is a triangle with base 3 and height $f(0) = 6$ (the line hits zero at $x = 3$).

$$\int_0^3 (6 - 2x)\,dx = \frac{1}{2}(3)(6) = 9$$

Average value:

$$f_{\text{ave}} = \frac{1}{3-0} \cdot 9 = \boxed{3}$$

Check: $f_{\text{ave}} = 3$ makes sense—it is the value of f at $x = 1.5$, the midpoint of $[0, 3]$, which is expected for a linear function.

8 Summary

Definite integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Geometric interpretation:

- If $f \geq 0$: equals area under the curve
- In general: net signed area ($= A_1 - A_2$)
- Total area: $\displaystyle\int_a^b |f(x)|\,dx = A_1 + A_2$

Key properties:

- Linearity: $\int c\,f \pm d\,g = c\int f \pm d\int g$
- Splitting: $\int_a^b f = \int_a^c f + \int_c^b f$
- Reversing limits: $\int_b^a f = -\int_a^b f$

Average value: $f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

Coming up (5.3–5.4): The Fundamental Theorem of Calculus—connecting derivatives and integrals—will give us a powerful shortcut that replaces limit calculations with antiderivatives.