The Fundamental Theorem of Calculus
Section 5.3
Calculus I - Lecture Notes

March 31, 2026

Motivating Question

Is there a faster way to compute a definite integral than summing thousands of rectangles?

Last class we defined $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$. We saw that this limit gave us $\frac{8}{3}$ for $f(x) = x^2$ on $[0, 2]$—but the algebra was painful. Today we uncover the most important theorem in all of calculus, one that makes nearly every definite integral easy to evaluate.

The idea is this: **differentiation and integration are inverse operations**. Finding the area under a curve is, in a precise sense, the reverse of finding a derivative. Newton and Leibniz discovered this connection independently in the late 1600s, and it changed the course of science and mathematics. By the end of today, you will be able to evaluate most definite integrals in seconds.

The Mean Value Theorem for Integrals

Before the Fundamental Theorem, we need one preliminary result about average values.

Definition 1. *The* ***average value*** *of f over $[a, b]$ is* $\bar{f} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx.$

Theorem 1 (Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then there exists at least one point $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = \bar{f}.$$

In plain English: *a continuous function always attains its own average value somewhere in the interval.* This is the integral version of the Mean Value Theorem from Chapter 4.

Example 1. *Find the average value of $f(x) = 8 - 2x$ on $[0, 4]$, then find c where $f(c)$ equals that average.*

Method: *Compute the integral, divide by $b - a$, then solve $f(c) = \bar{f}$.*

Solution:

The graph of $f(x) = 8 - 2x$ is a line from $(0, 8)$ to $(4, 0)$, forming a right triangle. The area is

$$\int_0^4 (8 - 2x)\,dx = \tfrac{1}{2}(4)(8) = 16.$$

Average value:

$$\bar{f} = \frac{1}{4} \cdot 16 = 4.$$

Solve $f(c) = 4$:

$$8 - 2c = 4 \implies c = 2.$$

Example 2. *Given $\int_0^3 x^2\,dx = 9$, find c such that $f(c) = \bar{f}$ for $f(x) = x^2$ on $[0, 3]$.*

Solution:

$$\bar{f} = \frac{1}{3 - 0} \cdot 9 = 3.$$

Set $f(c) = 3$:

$$c^2 = 3 \implies c = \pm\sqrt{3}.$$

Since $-\sqrt{3} \notin [0, 3]$, the answer is $c = \sqrt{3} \approx 1.73$.

0.1 Practice Problem

Given $\int_0^3 (2x^2 - 1)\,dx = 15$, find $c \in [0, 3]$ such that $f(c) = \bar{f}$ for $f(x) = 2x^2 - 1$.

Solution:

$$\bar{f} = \frac{1}{3} \cdot 15 = 5.$$

Set $f(c) = 5$:

$$2c^2 - 1 = 5 \implies c^2 = 3 \implies c = \sqrt{3}.$$

(We discard $c = -\sqrt{3}$ since it lies outside $[0, 3]$.) Answer: $c = \sqrt{3}$.

FTC Part 1: The Derivative of an Integral

1 The Accumulation Function

Define a new function by letting the upper limit of integration vary:

$$F(x) = \int_a^x f(t)\,dt.$$

Think of $F(x)$ as a running total: it measures the net area accumulated from a up to the current position x. We use t as the dummy variable inside the integral so it doesn't conflict with the upper limit x.

Theorem 2 (Fundamental Theorem of Calculus, Part 1)**.** *If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then*

$$F'(x) = f(x) \quad \textit{for all } x \in [a, b].$$

What this says: The *derivative* of the accumulated area equals the original function. Differentiation undoes integration—they are inverse operations.

Consequence: Every continuous function has an antiderivative. We can always write one down explicitly: $F(x) = \int_a^x f(t)\,dt$.

2 Using FTC Part 1

Pattern: To differentiate $\int_a^x f(t)\,dt$, just replace t with x in the integrand. No integration required.

Example 3. *Use FTC Part 1 to find the derivative of $g(x) = \int_1^x \frac{1}{t^3+1}\,dt$.*

Solution:

The integrand is $f(t) = \frac{1}{t^3+1}$, which is continuous on $[1, \infty)$. By FTC Part 1, just plug in x:

$$g'(x) = \frac{1}{x^3+1}.$$

Done—no integration needed.

2.1 Chain Rule Variant

When the upper limit is a function of x, we need the chain rule. The pattern is:

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = \underbrace{f(u(x))}_{\text{plug in upper limit}} \cdot \underbrace{u'(x)}_{\text{multiply by derivative of upper limit}}.$$

Example 4. *Let* $F(x) = \int_1^{\sqrt{x}} \sin t \, dt$*. Find* $F'(x)$*.*

Solution:
Upper limit $u(x) = \sqrt{x}$*, so* $u'(x) = \frac{1}{2\sqrt{x}}$*:*

$$F'(x) = \sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

Example 5. *Let* $F(x) = \int_1^{x^3} \cos t \, dt$*. Find* $F'(x)$*.*

Solution:
Upper limit $u(x) = x^3$*,* $u'(x) = 3x^2$*:*

$$F'(x) = \cos(x^3) \cdot 3x^2 = 3x^2 \cos(x^3).$$

2.2 Both Limits Variable

When *both* limits depend on x, first split the integral at any convenient constant a, then differentiate each piece separately:

$$\int_{u(x)}^{v(x)} f(t)\, dt = \int_a^{v(x)} f(t)\, dt - \int_a^{u(x)} f(t)\, dt.$$

Example 6. *Let* $F(x) = \int_x^{2x} t^3 \, dt$*. Find* $F'(x)$*.*

Solution:
Split at 0*:*

$$F(x) = \int_0^{2x} t^3\, dt - \int_0^{x} t^3\, dt.$$

Differentiate the first term with $u(x) = 2x$*,* $u'(x) = 2$*:*

$$\frac{d}{dx}\int_0^{2x} t^3\, dt = (2x)^3 \cdot 2 = 16x^3.$$

Differentiate the second term with $u(x) = x$*,* $u'(x) = 1$*:*

$$\frac{d}{dx}\int_0^{x} t^3\, dt = x^3 \cdot 1 = x^3.$$

Therefore:

$$F'(x) = 16x^3 - x^3 = 15x^3.$$

2.3 Practice Problem

Let $F(x) = \int_x^{x^2} \cos t\, dt$**. Find** $F'(x)$**.**

Solution:
Split at 0:

$$F(x) = \int_0^{x^2} \cos t\, dt - \int_0^{x} \cos t\, dt.$$

Differentiate:

$$\frac{d}{dx}\int_0^{x^2} \cos t\, dt = \cos(x^2) \cdot 2x = 2x\cos(x^2).$$

$$\frac{d}{dx}\int_0^{x} \cos t\, dt = \cos(x).$$

Therefore:

$$F'(x) = 2x\cos(x^2) - \cos x.$$

FTC Part 2: The Evaluation Theorem

3 The Evaluation Theorem

Part 1 tells us that integration and differentiation are inverses. Part 2 makes this operational: to evaluate a definite integral, just find an antiderivative.

Theorem 3 (Fundamental Theorem of Calculus, Part 2)**.** *If f is continuous on $[a, b]$ and F is any antiderivative of f (meaning $F'(x) = f(x)$), then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Notation: We write $F(x)\Big|_a^b$ to mean $F(b) - F(a)$.

Why it works: FTC Part 1 tells us $G(x) = \int_a^x f(t)\, dt$ is *an* antiderivative of f. Any other antiderivative F differs from G by a constant: $F(x) = G(x) + C$. When we compute $F(b) - F(a)$, that constant cancels:

$$F(b) - F(a) = [G(b) + C] - [G(a) + C] = G(b) - G(a) = \int_a^b f(t)\, dt.$$

Practical note: Always set $C = 0$ (drop the "$+C$") when evaluating a definite integral. It cancels regardless.

4 Computing Definite Integrals

Pattern: (1) Find an antiderivative F. (2) Evaluate $F(b) - F(a)$. That's it.

Example 7. *Evaluate* $\int_{-2}^{2} (t^2 - 4)\, dt$.

Solution:
An antiderivative of $t^2 - 4$ *is* $F(t) = \frac{t^3}{3} - 4t$.

$$\begin{aligned}\int_{-2}^{2} (t^2 - 4)\, dt &= \left(\frac{t^3}{3} - 4t\right)\Big|_{-2}^{2} \\ &= \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right) \\ &= \frac{8}{3} - 8 + \frac{8}{3} - 8 \\ &= \frac{16}{3} - 16 = -\frac{32}{3}.\end{aligned}$$

Note: *A negative result is perfectly fine—the parabola* $t^2 - 4$ *lies below the* x*-axis on* $(-2, 2)$*, so the signed area is negative.*

Example 8. *Evaluate* $\int_{1}^{9} \frac{x-1}{\sqrt{x}}\, dx$.

Solution:
Rewrite by splitting the fraction (using $\sqrt{x} = x^{1/2}$*):*

$$\frac{x-1}{x^{1/2}} = \frac{x}{x^{1/2}} - \frac{1}{x^{1/2}} = x^{1/2} - x^{-1/2}.$$

Now integrate using the power rule:

$$\begin{aligned}\int_{1}^{9} \left(x^{1/2} - x^{-1/2}\right) dx &= \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}}\right)\Big|_{1}^{9} \\ &= \left(\frac{2}{3}x^{3/2} - 2x^{1/2}\right)\Big|_{1}^{9} \\ &= \left(\frac{2}{3}(27) - 2(3)\right) - \left(\frac{2}{3}(1) - 2(1)\right) \\ &= (18 - 6) - \left(\frac{2}{3} - 2\right) \\ &= 12 - \left(-\frac{4}{3}\right) = 12 + \frac{4}{3} = \frac{40}{3}.\end{aligned}$$

Example 9 (Roller-Skating Race)**.** *James skates at $f(t) = 5 + 2t$ ft/sec and Kathy at $g(t) = 10 + \cos\left(\frac{\pi}{2}t\right)$ ft/sec. Who travels farther in 5 seconds?*

Solution:
James:

$$\int_0^5 (5 + 2t)\, dt = \left(5t + t^2\right)\Big|_0^5 = 25 + 25 = 50 \text{ ft.}$$

Kathy: *We need an antiderivative of* $\cos\left(\frac{\pi}{2}t\right)$. *Since* $\frac{d}{dt}\sin\left(\frac{\pi}{2}t\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)$, *we get* $\int \cos\left(\frac{\pi}{2}t\right) dt = \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)$.

$$\begin{aligned}\int_0^5 \left(10 + \cos\tfrac{\pi t}{2}\right) dt &= \left(10t + \frac{2}{\pi}\sin\frac{\pi t}{2}\right)\Bigg|_0^5 \\ &= \left(50 + \frac{2}{\pi}\sin\frac{5\pi}{2}\right) - (0 + 0) \\ &= 50 + \frac{2}{\pi}(1) \approx 50.64 \text{ ft.}\end{aligned}$$

Kathy wins*, but barely.*

4.1 Practice Problem

Evaluate $\int_1^2 x^{-4}\, dx$**.**

Solution:
An antiderivative of x^{-4} is $\frac{x^{-3}}{-3} = -\frac{1}{3x^3}$.

$$\int_1^2 x^{-4}\, dx = \left(-\frac{1}{3x^3}\right)\Bigg|_1^2 = \left(-\frac{1}{24}\right) - \left(-\frac{1}{3}\right) = -\frac{1}{24} + \frac{8}{24} = \frac{7}{24}.$$

5 Summary

Mean Value Theorem for Integrals: If f is continuous on $[a, b]$, then f equals its average value $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\, dx$ at some $c \in [a, b]$.

FTC Part 1: If f is continuous and $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = f(x)$.

- Direct: $\frac{d}{dx}\int_a^x f(t)\, dt = f(x)$.

- Chain rule: $\dfrac{d}{dx}\displaystyle\int_a^{u(x)} f(t)\,dt = f(u(x)) \cdot u'(x)$.
- Both limits variable: split at a constant, apply chain rule to each piece.

FTC Part 2 (Evaluation Theorem): If $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

The constant of integration always cancels. This converts a hard limit-of-sums problem into a simple antiderivative evaluation.

Big picture: Integration and differentiation are inverse operations. The FTC is why we spent weeks on antiderivatives—every antiderivative rule is simultaneously an integration rule.