

Math 1231 — Midterm 1 (Version A)

Limits and Derivatives Sections 2.1–2.5, 3.1–3.6, 3.8

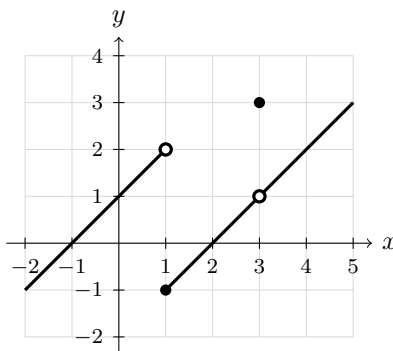
October 6, 2026 50 points

SOLUTIONS

Problem 1. Limits and continuity from a graph.

(6 points)

The graph of a function g is shown. Open circles are points *not* on the graph; filled dots are points on the graph.



(a) (2 points) Circle the value of $\lim_{x \rightarrow 3} g(x)$.

(A) 3

(B) does not exist

(C) 1 ←

(D) 2

Solution. Trace the graph toward $x = 3$ from both sides: the line heads for the open circle at height 1. The filled dot at height 3 is the *value* $g(3)$, and the limit ignores it. So $\lim_{x \rightarrow 3} g(x) = 1$. Answer: (C). Choice (A) is $g(3)$, the most common error.

(b) (2 points) Circle the true statement about g near $x = 1$.

(A) $\lim_{x \rightarrow 1} g(x) = 2$.

(B) $\lim_{x \rightarrow 1^-} g(x) = 2$ and $\lim_{x \rightarrow 1^+} g(x) = -1$, so $\lim_{x \rightarrow 1} g(x)$ does not exist. ←

(C) $\lim_{x \rightarrow 1} g(x) = -1$, because $g(1) = -1$.

(D) $\lim_{x \rightarrow 1^-} g(x) = -1$ and $\lim_{x \rightarrow 1^+} g(x) = 2$, so $\lim_{x \rightarrow 1} g(x)$ does not exist.

Solution. From the left the graph climbs to the open circle at height 2; from the right it starts at height -1 . The one-sided limits disagree, so the two-sided limit does not exist. Answer: (B). Choice (D) has the sides swapped; (C) confuses the value $g(1)$ with the limit.

(c) (2 points) For each of $x = 1$ and $x = 3$: is g continuous there? Explain why, using the three conditions for continuity. If g is not continuous, say which condition fails and classify the discontinuity as removable, jump, or infinite.

Solution. At $x = 1$: (i) $g(1) = -1$ is defined. (ii) The one-sided limits are 2 and -1 , so $\lim_{x \rightarrow 1} g(x)$ does not exist. Condition (ii) fails; this is a **jump discontinuity**.

At $x = 3$: (i) $g(3) = 3$ is defined. (ii) $\lim_{x \rightarrow 3} g(x) = 1$ exists. (iii) $1 \neq 3$, so condition (iii) fails; this is a **removable discontinuity**. Redefining $g(3) = 1$ would fill the hole.

Grading: 1 point for each of $x = 1$ and $x = 3$, each needing both the failed condition and the correct type.

Problem 2. Computing limits.

(8 points)

(a) (2 points) Circle the value of $\lim_{x \rightarrow 2} (x^2 + 3x - 1)$.

- (A) 3 (B) $9 \leftarrow$ (C) 10 (D) does not exist

Solution. A polynomial is continuous everywhere, so the limit is found by direct substitution: $2^2 + 3(2) - 1 = 4 + 6 - 1 = 9$. Answer: **(B)**.

(b) (2 points) Circle the value of $\lim_{x \rightarrow -2} \frac{x^2 + 5x + 6}{x^2 - 4}$.

- (A) $-\frac{1}{4} \leftarrow$ (B) $\frac{1}{4}$ (C) 0 (D) does not exist

Solution. Substituting $x = -2$ gives $\frac{4-10+6}{4-4} = \frac{0}{0}$, so factor and cancel:

$$\frac{x^2 + 5x + 6}{x^2 - 4} = \frac{(x+2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x-2} \quad (x \neq -2), \quad \lim_{x \rightarrow -2} \frac{x+3}{x-2} = \frac{1}{-4} = -\frac{1}{4}.$$

Answer: **(A)**. Choice (D) stops at $\frac{0}{0}$; choice (B) drops the sign of $-2 - 2$.

(c) (2 points) Circle the value of $\lim_{x \rightarrow 3^-} \frac{x+1}{x-3}$.

- (A) $+\infty$ (B) 0 (C) 1 (D) $-\infty \leftarrow$

Solution. Substituting gives $\frac{4}{0}$: a nonzero number over something approaching 0, so the limit is infinite and only the sign is in question. As $x \rightarrow 3^-$ we have $x < 3$, so the denominator $x - 3$ is a small *negative* number, while the numerator approaches $4 > 0$. Positive over tiny negative is hugely negative:

$$\lim_{x \rightarrow 3^-} \frac{x+1}{x-3} = -\infty,$$

and $x = 3$ is a vertical asymptote. Answer: **(D)**. Choice (A) misses the sign of the denominator from the left; (B) treats $\frac{4}{0}$ as 0; (C) cancels the x 's.

(d) (2 points) Evaluate $\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x+1}-2}$. Show all work.

Solution. Substituting gives $\frac{0}{\sqrt{4}-2} = \frac{0}{0}$. The square root is in the *denominator*, so multiply top and bottom by its conjugate $\sqrt{x+1}+2$:

$$\frac{x-3}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \frac{(x-3)(\sqrt{x+1}+2)}{(x+1)-4} = \frac{(x-3)(\sqrt{x+1}+2)}{x-3} = \sqrt{x+1}+2 \quad (x \neq 3).$$

Now substitute: $\lim_{x \rightarrow 3} (\sqrt{x+1}+2) = \sqrt{4}+2 = 4$.

Grading: 1 point for the conjugate step reaching a cancellable form, 1 point for cancelling and getting 4.

Problem 3. Continuity.

(6 points)

(a) (2 points) Let $f(x) = \begin{cases} kx + 1, & x < 2, \\ x^2 + 1, & x \geq 2. \end{cases}$ Circle the value of the constant k that makes f continuous at $x = 2$.

- (A) $k = 3$ (B) $k = 2 \leftarrow$ (C) $k = \frac{5}{2}$ (D) no value of k works

Solution. $f(2) = 4 + 1 = 5$ and $\lim_{x \rightarrow 2^+} f(x) = 5$. For the two-sided limit to exist and equal $f(2)$, the left-hand limit must match:

$$\lim_{x \rightarrow 2^-} f(x) = 2k + 1 = 5 \implies 2k = 4 \implies k = 2.$$

Check: both sides give 5. Answer: **(B)**. Choice (A) comes from writing $2k - 1$ on the left; (C) from dropping the $+1$.

(b) (2 points) Let $f(x) = \frac{x}{|x|}$. Circle the correct description of f at $x = 0$.

- (A) removable discontinuity, since $\lim_{x \rightarrow 0} f(x) = 1$ (B) removable discontinuity, since $\lim_{x \rightarrow 0} f(x) = -1$
 (C) jump discontinuity, with $\lim_{x \rightarrow 0^-} f(x) = 1$ and $\lim_{x \rightarrow 0^+} f(x) = -1$ (D) infinite discontinuity, since the denominator approaches 0
 (E) **jump discontinuity, with** $\lim_{x \rightarrow 0^-} f(x) = -1$ **and** $\lim_{x \rightarrow 0^+} f(x) = 1$ (F) continuous, since $f(x)$ simplifies to 1
 $\leftarrow$

Solution. $f(0)$ is undefined, so f is discontinuous at 0; the question is which type. The absolute value behaves differently on each side, so split into one-sided limits. For $x < 0$, $|x| = -x$; for $x > 0$, $|x| = x$:

$$\lim_{x \rightarrow 0^-} \frac{x}{-x} = \lim_{x \rightarrow 0^-} (-1) = -1, \quad \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1.$$

Both one-sided limits exist but disagree: a **jump discontinuity**. Answer: **(E)**.

The trap is (A) and (F): cancelling x against $|x|$ as if they were equal, which is only true for $x > 0$. (B) keeps only the left-hand value -1 . (C) has the sides swapped. (D) sees a zero denominator but misses that the numerator also goes to 0.

(c) (2 points) Use the Intermediate Value Theorem to show that the equation $x^3 + x - 3 = 0$ has a solution in the interval $[1, 2]$. State clearly why the theorem applies.

Solution. Let $f(x) = x^3 + x - 3$. As a polynomial, f is continuous everywhere, so in particular it is continuous on $[1, 2]$. Evaluate at the endpoints:

$$f(1) = 1 + 1 - 3 = -1 < 0, \quad f(2) = 8 + 2 - 3 = 7 > 0.$$

Since 0 lies between $f(1) = -1$ and $f(2) = 7$, the Intermediate Value Theorem gives a number c in $[1, 2]$ with $f(c) = 0$, i.e. a solution of $x^3 + x - 3 = 0$.

Grading: 1 point for stating continuity on $[1, 2]$; 1 point for both endpoint values with opposite signs and the conclusion. Omitting continuity loses a point: the theorem does not apply without it.

Problem 4. The precise definition of a limit.

(6 points)

(a) (2 points) In an epsilon-delta proof that $\lim_{x \rightarrow 2} (5x - 3) = 7$, circle a choice of δ that works for every $\varepsilon > 0$.

- (A) $\delta = 5\varepsilon$ (B) $\delta = \varepsilon$ (C) $\delta = \varepsilon + 5$ (D) $\delta = \frac{\varepsilon}{5} \leftarrow$

Solution. $|(5x - 3) - 7| = |5x - 10| = 5|x - 2|$. To force $5|x - 2| < \varepsilon$ we need $|x - 2| < \frac{\varepsilon}{5}$, so $\delta = \frac{\varepsilon}{5}$. Answer: **(D)**. Choice (A) inverts the division; (B) forgets the factor of 5 from the slope, so $|f(x) - 7|$ could be as large as 5ε ; (C) makes the window *wider* as ε shrinks toward 0, which can never work.

(b) (2 points) Circle the statement that is the precise definition of $\lim_{x \rightarrow a} f(x) = L$.

- (A) For every $\delta > 0$ there exists $\varepsilon > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
 (B) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f(x) - L| < \varepsilon$, then $0 < |x - a| < \delta$.
 (C) **For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.** $\leftarrow$
 (D) There exists $\varepsilon > 0$ such that for every $\delta > 0$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Solution. Answer: **(C)**. The skeptic names the output tolerance ε (*for every*), and we must answer with an input window δ (*there exists*) that keeps $f(x)$ within ε of L .

- **(A)** swaps the roles: any bounded function satisfies it for a large enough ε , so it says nothing about L .
- **(B)** reverses the implication: it controls x given $f(x)$, not the other way around.
- **(D)** allows one fixed, possibly huge, tolerance instead of every tolerance.

(c) (2 points) Use the epsilon-delta definition to prove that $\lim_{x \rightarrow -2} (3x + 4) = -2$. Include your scratch work for finding δ , then write the proof.

Solution. *Scratch work.* We want $|(3x + 4) - (-2)| < \varepsilon$. Simplify:

$$|(3x + 4) - (-2)| = |3x + 6| = 3|x + 2| = 3|x - (-2)|.$$

So we need $|x - (-2)| < \frac{\varepsilon}{3}$. Choose $\delta = \frac{\varepsilon}{3}$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{3}$. Assume $0 < |x - (-2)| < \delta$. Then

$$|(3x + 4) - (-2)| = |3x + 6| = 3|x + 2| < 3\delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Therefore $\lim_{x \rightarrow -2} (3x + 4) = -2$. ■

Grading: 1 point for the scratch work reaching $\delta = \frac{\varepsilon}{3}$; 1 point for a proof with all five steps (let ε , choose δ , assume, chain of inequalities, conclude).

Problem 5. The derivative as a limit.

(6 points)

(a) (2 points) Circle the value of $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$.

- (A) 6 ← (B) 9 (C) 0 (D) does not exist

Solution. Since $9 = 3^2$, this is $\lim_{h \rightarrow 0} \frac{f(3+h)-f(3)}{h}$ for $f(x) = x^2$: the definition of $f'(3)$. By the power rule $f'(x) = 2x$, so $f'(3) = 6$. (Expanding works too: $\frac{9+6h+h^2-9}{h} = 6+h \rightarrow 6$.) Answer: **(A)**. Choice (B) is $f(3)$, the value rather than the slope; (C) and (D) stop at $\frac{0}{0}$.

(b) (2 points) Let $f(x) = \begin{cases} x^2, & x \leq 1, \\ 2x + b, & x > 1, \end{cases}$ where b is a constant. Circle the value of b that makes f differentiable at $x = 1$.

- (A) $b = 1$ (B) $b = 3$ (C) $b = -1$ ← (D) no value of b works

Solution. Two requirements. **Matching slopes:** from the left the slope is that of x^2 at 1, namely $2(1) = 2$; from the right it is the slope of the line, also 2. The slopes agree for every b , so continuity decides it. **Continuity**, since differentiability implies continuity: the pieces must meet at $x = 1$, so $1^2 = 2(1) + b$, i.e. $b = -1$. Answer: **(C)**. The function is x^2 glued to its own tangent line $y = 2x - 1$ at $x = 1$.

(B) comes from $b = 1 + 2$ instead of $1 - 2$. (A) copies the value $f(1) = 1$. (D) comes from thinking a line cannot match the slope of a parabola.

(c) (2 points) Let $f(x) = \frac{1}{x}$. Find $f'(x)$ using the definition of the derivative as a limit. *No derivative rules.* Show each step.

Solution. Step 1. $f(x+h) = \frac{1}{x+h}$.

Step 2. Combine the difference of fractions over a common denominator:

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{-h}{x(x+h)}.$$

Step 3. Divide by h and cancel:

$$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x(x+h)}.$$

Step 4. Take the limit:

$$f'(x) = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

Grading: 1 point for a correct common-denominator difference quotient; 1 point for cancelling h and taking the limit.

Problem 6. Differentiation rules and rates of change.

(6 points)

(a) (2 points) Suppose $f(1) = 2$, $f'(1) = -3$, $g(1) = 4$, and $g'(1) = 5$. Let $q(x) = \frac{f(x)}{g(x)}$. Circle $q'(1)$.

- (A) $\frac{11}{8}$ (B) $-\frac{3}{5}$ (C) $-\frac{1}{8}$ (D) $-\frac{11}{8} \leftarrow$

Solution. The quotient rule: derivative of the top times the bottom, minus the top times the derivative of the bottom, over the bottom squared.

$$q'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \frac{(-3)(4) - (2)(5)}{4^2} = \frac{-12 - 10}{16} = -\frac{22}{16} = -\frac{11}{8}.$$

Answer: **(D)**. Choice (A) writes the numerator backwards, flipping the sign; (B) is $\frac{f'(1)}{g'(1)}$, the derivative of the top over the derivative of the bottom; (C) uses $+$ in the numerator.

(b) (2 points) A particle's position at time t seconds is $s(t) = t^2 - 6t + 5$ meters. At $t = 2$, the particle is:

- (A) moving in the positive direction and speeding up (B) **moving in the negative direction and slowing down** $\leftarrow$
(C) moving in the negative direction and speeding up (D) moving in the positive direction and slowing down

Solution. $v(t) = s'(t) = 2t - 6$ and $a(t) = v'(t) = 2$. At $t = 2$: $v(2) = -2 < 0$, so the particle moves in the negative direction. $a(2) = 2 > 0$ has the opposite sign from v , so the acceleration fights the motion and the particle is slowing down. Answer: **(B)**. (The sign of the position, $s(2) = -3$, says where the particle *is*, not which way it is going.)

(c) (2 points) Let $f(x) = x \cos x$. Find an equation of the line tangent to the graph of f at $x = \pi$.

Solution. The point. $f(\pi) = \pi \cos \pi = -\pi$, so the point is $(\pi, -\pi)$.

The slope. By the product rule,

$$f'(x) = (1) \cos x + x(-\sin x) = \cos x - x \sin x, \quad f'(\pi) = \cos \pi - \pi \sin \pi = -1 - 0 = -1.$$

Assemble. Point-slope form: $y - (-\pi) = -1(x - \pi)$, which simplifies to $y = -x$.

Grading: 1 point for $f'(x)$ by the product rule; 1 point for the correct point, slope, and line.

Problem 7. The chain rule.

(6 points)

(a) (2 points) Let $h(x) = \cos^3(2x)$. Circle $h'\left(\frac{\pi}{6}\right)$.

- (A) $-\frac{3\sqrt{3}}{8}$ (B) $\frac{3\sqrt{3}}{4}$ (C) $-\frac{3\sqrt{3}}{4} \leftarrow$ (D) $-\frac{9}{4}$

Solution. Three layers: the cube, then cosine, then $2x$. Peel them from the outside in:

$$h'(x) = 3\cos^2(2x) \cdot (-\sin(2x)) \cdot 2 = -6\cos^2(2x)\sin(2x).$$

At $x = \frac{\pi}{6}$ the inside is $2x = \frac{\pi}{3}$, where $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$:

$$h'\left(\frac{\pi}{6}\right) = -6\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{4}.$$

Answer: **(C)**. Choice (A) forgets the innermost factor 2; (B) differentiates $\cos$ as $+\sin$; (D) evaluates the trig functions at $\frac{\pi}{6}$ instead of at $2x = \frac{\pi}{3}$.**(b)** (2 points) Use the table. Let $h(x) = f(g(x))$. Circle $h'(2)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	1	4	3	-1
3	7	5	0	6

- (A) $-5 \leftarrow$ (B) -4 (C) 5 (D) 15

Solution. By the chain rule, $h'(2) = f'(g(2))g'(2)$. From the table, $g(2) = 3$, so

$$h'(2) = f'(3) \cdot g'(2) = 5 \cdot (-1) = -5.$$

Answer: **(A)**. Choice (B) evaluates f' at 2 instead of at $g(2) = 3$; (C) forgets the factor $g'(2)$; (D) multiplies by $g(2)$ instead of $g'(2)$.**(c)** (2 points) Find the derivative of $k(x) = \tan^4(3x^2 + 1)$.**Solution.** Rewrite to see the layers: $k(x) = \left(\tan(3x^2 + 1)\right)^4$. Outside to inside: the 4th power, then tangent, then $3x^2 + 1$. Peel one layer at a time:

$$\begin{aligned}
 k'(x) &= 4\left(\tan(3x^2 + 1)\right)^3 \cdot \frac{d}{dx}\left[\tan(3x^2 + 1)\right] && \text{power layer} \\
 &= 4\tan^3(3x^2 + 1) \cdot \sec^2(3x^2 + 1) \cdot 6x && \text{tangent layer, then the inside} \\
 &= 24x \tan^3(3x^2 + 1) \sec^2(3x^2 + 1).
 \end{aligned}$$

Grading: 1 point for the power and tangent layers (including $\frac{d}{dx} \tan u = \sec^2 u$); 1 point for the inner factor $6x$ and a correct final answer.

Problem 8. Implicit differentiation.

(6 points)

(a) (2 points) Assuming y is defined implicitly by $x^2 + 4y^2 = 20$, circle $\frac{dy}{dx}$.

- (A) $\frac{x}{4y}$ (B) $-\frac{x}{4y} \leftarrow$ (C) $-\frac{4y}{x}$ (D) $-\frac{x}{2y}$

Solution. Differentiate both sides with respect to x ; the chain rule gives $\frac{d}{dx}(4y^2) = 8y \frac{dy}{dx}$:

$$2x + 8y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{2x}{8y} = -\frac{x}{4y}.$$

Answer: **(B)**. Choice (D) differentiates $4y^2$ as $4y \frac{dy}{dx}$; (C) is the reciprocal; (A) loses the sign when moving $2x$ across.

(b) (2 points) The point $(1, 2)$ lies on the curve $x^2 + xy + y^2 = 7$. Circle the slope of the line tangent to the curve at $(1, 2)$.

- (A) -1 (B) $-\frac{2}{5}$ (C) $\frac{4}{5}$ (D) $-\frac{4}{5} \leftarrow$

Solution. Differentiate; xy needs the product rule and y^2 the chain rule:

$$2x + \left(y + x \frac{dy}{dx}\right) + 2y \frac{dy}{dx} = 0 \implies \frac{dy}{dx}(x + 2y) = -(2x + y) \implies \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

At $(1, 2)$: $\frac{dy}{dx} = -\frac{2+2}{1+4} = -\frac{4}{5}$. Answer: **(D)**.

Choice (A) differentiates xy as just y , giving $-\frac{2x+y}{2y} = -1$; (B) differentiates it as just $x \frac{dy}{dx}$, giving $-\frac{2x}{x+2y} = -\frac{2}{5}$; (C) drops the sign.

(c) (2 points) Assuming y is defined implicitly by $x^2y^3 + \cos y = 5x$, find $\frac{dy}{dx}$.

Solution. Differentiate both sides. The term x^2y^3 needs the product rule, with the chain rule on y^3 ; and $\frac{d}{dx}(\cos y) = -\sin y \frac{dy}{dx}$:

$$\underbrace{2x y^3 + x^2 \cdot 3y^2 \frac{dy}{dx}}_{\text{product rule on } x^2y^3} - \sin y \frac{dy}{dx} = 5.$$

Collect the $\frac{dy}{dx}$ terms on the left, factor, and divide:

$$\frac{dy}{dx}(3x^2y^2 - \sin y) = 5 - 2xy^3 \implies \frac{dy}{dx} = \frac{5 - 2xy^3}{3x^2y^2 - \sin y}.$$

Grading: 1 point for correctly differentiating every term (product rule, and a $\frac{dy}{dx}$ on each y -term); 1 point for solving for $\frac{dy}{dx}$.

Math 1231 — Midterm 1 (Version B)

Limits and Derivatives Sections 2.1–2.5, 3.1–3.6, 3.8

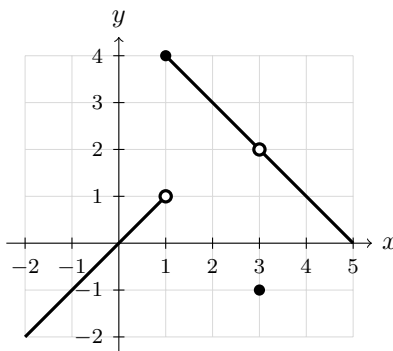
October 6, 2026 50 points

SOLUTIONS

Problem 1. Limits and continuity from a graph.

(6 points)

The graph of a function g is shown. Open circles are points *not* on the graph; filled dots are points on the graph.



(a) (2 points) Circle the value of $\lim_{x \rightarrow 3} g(x)$.

(A) 2 ←

(B) -1

(C) 0

(D) does not exist

Solution. Trace the graph toward $x = 3$ from both sides: the line heads for the open circle at height 2. The filled dot at height -1 is the *value* $g(3)$, and the limit ignores it. So $\lim_{x \rightarrow 3} g(x) = 2$. Answer: (A). Choice (B) is $g(3)$, the most common error.

(b) (2 points) Circle the true statement about g near $x = 1$.

(A) $\lim_{x \rightarrow 1} g(x) = 4$, because $g(1) = 4$.

(B) $\lim_{x \rightarrow 1} g(x) = 1$.

(C) $\lim_{x \rightarrow 1^-} g(x) = 4$ and $\lim_{x \rightarrow 1^+} g(x) = 1$, so $\lim_{x \rightarrow 1} g(x)$ does not exist.

(D) $\lim_{x \rightarrow 1^-} g(x) = 1$ **and** $\lim_{x \rightarrow 1^+} g(x) = 4$, **so** $\lim_{x \rightarrow 1} g(x)$ **does not exist**. ←

Solution. From the left the graph climbs to the open circle at height 1; from the right it starts at height 4. The one-sided limits disagree, so the two-sided limit does not exist. Answer: (D). Choice (C) has the sides swapped; (A) confuses the value $g(1)$ with the limit.

(c) (2 points) For each of $x = 1$ and $x = 3$: is g continuous there? Explain why, using the three conditions for continuity. If g is not continuous, say which condition fails and classify the discontinuity as removable, jump, or infinite.

Solution. At $x = 1$: (i) $g(1) = 4$ is defined. (ii) The one-sided limits are 1 and 4, so $\lim_{x \rightarrow 1} g(x)$ does not exist. Condition (ii) fails; this is a **jump discontinuity**.

At $x = 3$: (i) $g(3) = -1$ is defined. (ii) $\lim_{x \rightarrow 3} g(x) = 2$ exists. (iii) $2 \neq -1$, so condition (iii) fails; this is a **removable discontinuity**. Redefining $g(3) = 2$ would fill the hole.

Grading: 1 point for each of $x = 1$ and $x = 3$, each needing both the failed condition and the correct type.

Problem 2. Computing limits.

(8 points)

(a) (2 points) Circle the value of $\lim_{x \rightarrow 2} (x^2 + 3x - 1)$.

- (A) 10 (B) 3 (C) does not exist (D) 9 ←

Solution. A polynomial is continuous everywhere, so the limit is found by direct substitution: $2^2 + 3(2) - 1 = 4 + 6 - 1 = 9$. Answer: **(D)**.**(b)** (2 points) Circle the value of $\lim_{x \rightarrow -2} \frac{x^2 + 5x + 6}{x^2 - 4}$.

- (A) 0 (B)
- $\frac{1}{4}$
- (C)
- $-\frac{1}{4}$
- ← (D) does not exist

Solution. Substituting $x = -2$ gives $\frac{4-10+6}{4-4} = \frac{0}{0}$, so factor and cancel:

$$\frac{x^2 + 5x + 6}{x^2 - 4} = \frac{(x+2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x-2} \quad (x \neq -2), \quad \lim_{x \rightarrow -2} \frac{x+3}{x-2} = \frac{1}{-4} = -\frac{1}{4}.$$

Answer: **(C)**. Choice (D) stops at $\frac{0}{0}$; choice (B) drops the sign of $-2 - 2$.**(c)** (2 points) Circle the value of $\lim_{x \rightarrow 3^-} \frac{x+1}{x-3}$.

- (A)
- $-\infty$
- ← (B) 1 (C)
- $+\infty$
- (D) 0

Solution. Substituting gives $\frac{4}{0}$: a nonzero number over something approaching 0, so the limit is infinite and only the sign is in question. As $x \rightarrow 3^-$ we have $x < 3$, so the denominator $x - 3$ is a small *negative* number, while the numerator approaches $4 > 0$. Positive over tiny negative is hugely negative:

$$\lim_{x \rightarrow 3^-} \frac{x+1}{x-3} = -\infty,$$

and $x = 3$ is a vertical asymptote. Answer: **(A)**. Choice (C) misses the sign of the denominator from the left; (D) treats $\frac{4}{0}$ as 0; (B) cancels the x 's.**(d)** (2 points) Evaluate $\lim_{x \rightarrow 5} \frac{x-5}{\sqrt{x+4}-3}$. Show all work.**Solution.** Substituting gives $\frac{0}{\sqrt{9}-3} = \frac{0}{0}$. The square root is in the *denominator*, so multiply top and bottom by its conjugate $\sqrt{x+4}+3$:

$$\frac{x-5}{\sqrt{x+4}-3} \cdot \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = \frac{(x-5)(\sqrt{x+4}+3)}{(x+4)-9} = \frac{(x-5)(\sqrt{x+4}+3)}{x-5} = \sqrt{x+4}+3 \quad (x \neq 5).$$

Now substitute: $\lim_{x \rightarrow 5} (\sqrt{x+4}+3) = \sqrt{9}+3 = 6$.**Grading:** 1 point for the conjugate step reaching a cancellable form, 1 point for cancelling and getting 6.

Problem 3. Continuity.

(6 points)

(a) (2 points) Let $f(x) = \begin{cases} kx + 1, & x < 2, \\ x^2 + 1, & x \geq 2. \end{cases}$ Circle the value of the constant k that makes f continuous at $x = 2$.

- (A) $k = \frac{5}{2}$ (B) $k = 3$ (C) $k = 2 \leftarrow$ (D) no value of k works

Solution. $f(2) = 4 + 1 = 5$ and $\lim_{x \rightarrow 2^+} f(x) = 5$. For the two-sided limit to exist and equal $f(2)$, the left-hand limit must match:

$$\lim_{x \rightarrow 2^-} f(x) = 2k + 1 = 5 \implies 2k = 4 \implies k = 2.$$

Check: both sides give 5. Answer: **(C)**. Choice (B) comes from writing $2k - 1$ on the left; (A) from dropping the $+1$.

(b) (2 points) Let $f(x) = \frac{x}{|x|}$. Circle the correct description of f at $x = 0$.

- (A) infinite discontinuity, since the denominator approaches 0 (B) continuous, since $f(x)$ simplifies to 1
- (C) **jump discontinuity, with** (D) removable discontinuity, since
 $\lim_{x \rightarrow 0^-} f(x) = -1$ and $\lim_{x \rightarrow 0^+} f(x) = 1$ $\lim_{x \rightarrow 0} f(x) = 1$
 $\leftarrow$
- (E) jump discontinuity, with $\lim_{x \rightarrow 0^-} f(x) = 1$ (F) removable discontinuity, since
and $\lim_{x \rightarrow 0^+} f(x) = -1$ $\lim_{x \rightarrow 0} f(x) = -1$

Solution. $f(0)$ is undefined, so f is discontinuous at 0; the question is which type. The absolute value behaves differently on each side, so split into one-sided limits. For $x < 0$, $|x| = -x$; for $x > 0$, $|x| = x$:

$$\lim_{x \rightarrow 0^-} \frac{x}{-x} = \lim_{x \rightarrow 0^-} (-1) = -1, \quad \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1.$$

Both one-sided limits exist but disagree: a **jump discontinuity**. Answer: **(C)**.

The trap is (D) and (B): cancelling x against $|x|$ as if they were equal, which is only true for $x > 0$. (F) keeps only the left-hand value -1 . (E) has the sides swapped. (A) sees a zero denominator but misses that the numerator also goes to 0.

(c) (2 points) Use the Intermediate Value Theorem to show that the equation $x^3 + 2x - 5 = 0$ has a solution in the interval $[1, 2]$. State clearly why the theorem applies.

Solution. Let $f(x) = x^3 + 2x - 5$. As a polynomial, f is continuous everywhere, so in particular it is continuous on $[1, 2]$. Evaluate at the endpoints:

$$f(1) = 1 + 2 - 5 = -2 < 0, \quad f(2) = 8 + 4 - 5 = 7 > 0.$$

Since 0 lies between $f(1) = -2$ and $f(2) = 7$, the Intermediate Value Theorem gives a number c in $[1, 2]$ with $f(c) = 0$, i.e. a solution of $x^3 + 2x - 5 = 0$.

Grading: 1 point for stating continuity on $[1, 2]$; 1 point for both endpoint values with opposite signs and the conclusion. Omitting continuity loses a point: the theorem does not apply without it.

Problem 4. The precise definition of a limit.

(6 points)

(a) (2 points) In an epsilon-delta proof that $\lim_{x \rightarrow 2} (5x - 3) = 7$, circle a choice of δ that works for every $\varepsilon > 0$.

- (A) $\delta = \varepsilon$ (B) $\delta = \frac{\varepsilon}{5} \leftarrow$ (C) $\delta = 5\varepsilon$ (D) $\delta = \varepsilon + 5$

Solution. $|(5x - 3) - 7| = |5x - 10| = 5|x - 2|$. To force $5|x - 2| < \varepsilon$ we need $|x - 2| < \frac{\varepsilon}{5}$, so $\delta = \frac{\varepsilon}{5}$. Answer: **(B)**. Choice (C) inverts the division; (A) forgets the factor of 5 from the slope, so $|f(x) - 7|$ could be as large as 5ε ; (D) makes the window *wider* as ε shrinks toward 0, which can never work.

(b) (2 points) Circle the statement that is the precise definition of $\lim_{x \rightarrow a} f(x) = L$.

- (A) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f(x) - L| < \varepsilon$, then $0 < |x - a| < \delta$.
 (B) There exists $\varepsilon > 0$ such that for every $\delta > 0$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
 (C) For every $\delta > 0$ there exists $\varepsilon > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
 (D) **For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.** $\leftarrow$

Solution. Answer: **(D)**. The skeptic names the output tolerance ε (*for every*), and we must answer with an input window δ (*there exists*) that keeps $f(x)$ within ε of L .

- **(C)** swaps the roles: any bounded function satisfies it for a large enough ε , so it says nothing about L .
- **(A)** reverses the implication: it controls x given $f(x)$, not the other way around.
- **(B)** allows one fixed, possibly huge, tolerance instead of every tolerance.

(c) (2 points) Use the epsilon-delta definition to prove that $\lim_{x \rightarrow -1} (3x + 5) = 2$. Include your scratch work for finding δ , then write the proof.

Solution. *Scratch work.* We want $|(3x + 5) - 2| < \varepsilon$. Simplify:

$$|(3x + 5) - 2| = |3x + 3| = 3|x + 1| = 3|x - (-1)|.$$

So we need $|x - (-1)| < \frac{\varepsilon}{3}$. Choose $\delta = \frac{\varepsilon}{3}$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{3}$. Assume $0 < |x - (-1)| < \delta$. Then

$$|(3x + 5) - 2| = |3x + 3| = 3|x + 1| < 3\delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Therefore $\lim_{x \rightarrow -1} (3x + 5) = 2$. ■

Grading: 1 point for the scratch work reaching $\delta = \frac{\varepsilon}{3}$; 1 point for a proof with all five steps (let ε , choose δ , assume, chain of inequalities, conclude).

Problem 5. The derivative as a limit.

(6 points)

(a) (2 points) Circle the value of $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$.

- (A) 0 (B) $6 \leftarrow$ (C) does not exist (D) 9

Solution. Since $9 = 3^2$, this is $\lim_{h \rightarrow 0} \frac{f(3+h)-f(3)}{h}$ for $f(x) = x^2$: the definition of $f'(3)$. By the power rule $f'(x) = 2x$, so $f'(3) = 6$. (Expanding works too: $\frac{9+6h+h^2-9}{h} = 6+h \rightarrow 6$.) Answer: **(B)**. Choice (D) is $f(3)$, the value rather than the slope; (A) and (C) stop at $\frac{0}{0}$.

(b) (2 points) Let $f(x) = \begin{cases} x^2, & x \leq 1, \\ 2x + b, & x > 1, \end{cases}$ where b is a constant. Circle the value of b that makes f differentiable at $x = 1$.

- (A) $b = -1 \leftarrow$ (B) no value of b works (C) $b = 1$ (D) $b = 3$

Solution. Two requirements. **Matching slopes:** from the left the slope is that of x^2 at 1, namely $2(1) = 2$; from the right it is the slope of the line, also 2. The slopes agree for every b , so continuity decides it. **Continuity**, since differentiability implies continuity: the pieces must meet at $x = 1$, so $1^2 = 2(1) + b$, i.e. $b = -1$. Answer: **(A)**. The function is x^2 glued to its own tangent line $y = 2x - 1$ at $x = 1$.

(D) comes from $b = 1 + 2$ instead of $1 - 2$. (C) copies the value $f(1) = 1$. (B) comes from thinking a line cannot match the slope of a parabola.

(c) (2 points) Let $f(x) = \frac{2}{x}$. Find $f'(x)$ using the definition of the derivative as a limit. *No derivative rules.* Show each step.

Solution. Step 1. $f(x+h) = \frac{2}{x+h}$.

Step 2. Combine the difference of fractions over a common denominator:

$$f(x+h) - f(x) = \frac{2}{x+h} - \frac{2}{x} = \frac{2x - 2(x+h)}{x(x+h)} = \frac{-2h}{x(x+h)}.$$

Step 3. Divide by h and cancel:

$$\frac{f(x+h) - f(x)}{h} = \frac{-2}{x(x+h)}.$$

Step 4. Take the limit:

$$f'(x) = \lim_{h \rightarrow 0} \frac{-2}{x(x+h)} = -\frac{2}{x^2}.$$

Grading: 1 point for a correct common-denominator difference quotient; 1 point for cancelling h and taking the limit.

Problem 6. Differentiation rules and rates of change.

(6 points)

(a) (2 points) Suppose $f(1) = 2$, $f'(1) = -3$, $g(1) = 4$, and $g'(1) = 5$. Let $q(x) = \frac{f(x)}{g(x)}$. Circle $q'(1)$.

(A) $-\frac{1}{8}$

(B) $\frac{11}{8}$

(C) $-\frac{11}{8} \leftarrow$

(D) $-\frac{3}{5}$

Solution. The quotient rule: derivative of the top times the bottom, minus the top times the derivative of the bottom, over the bottom squared.

$$q'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \frac{(-3)(4) - (2)(5)}{4^2} = \frac{-12 - 10}{16} = -\frac{22}{16} = -\frac{11}{8}.$$

Answer: **(C)**. Choice (B) writes the numerator backwards, flipping the sign; (D) is $\frac{f'(1)}{g'(1)}$, the derivative of the top over the derivative of the bottom; (A) uses $+$ in the numerator.

(b) (2 points) A particle's position at time t seconds is $s(t) = t^2 - 6t + 5$ meters. At $t = 2$, the particle is:

(A) **moving in the negative direction and slowing down** $\leftarrow$ (B) moving in the positive direction and slowing down

(C) moving in the negative direction and speeding up (D) moving in the positive direction and speeding up

Solution. $v(t) = s'(t) = 2t - 6$ and $a(t) = v'(t) = 2$. At $t = 2$: $v(2) = -2 < 0$, so the particle moves in the negative direction. $a(2) = 2 > 0$ has the opposite sign from v , so the acceleration fights the motion and the particle is slowing down. Answer: **(A)**. (The sign of the position, $s(2) = -3$, says where the particle is, not which way it is going.)

(c) (2 points) Let $f(x) = 2x \cos x$. Find an equation of the line tangent to the graph of f at $x = \pi$.

Solution. The point. $f(\pi) = 2\pi \cos \pi = -2\pi$, so the point is $(\pi, -2\pi)$.

The slope. By the product rule,

$$f'(x) = (2) \cos x + 2x(-\sin x) = 2 \cos x - 2x \sin x, \quad f'(\pi) = 2 \cos \pi - 2\pi \sin \pi = -2 - 0 = -2.$$

Assemble. Point-slope form: $y - (-2\pi) = -2(x - \pi)$, which simplifies to $y = -2x$.

Grading: 1 point for $f'(x)$ by the product rule; 1 point for the correct point, slope, and line.

Problem 7. The chain rule.

(6 points)

(a) (2 points) Let $h(x) = \cos^3(2x)$. Circle $h'\left(\frac{\pi}{6}\right)$.

- (A) $\frac{3\sqrt{3}}{4}$ (B) $-\frac{9}{4}$ (C) $-\frac{3\sqrt{3}}{8}$ (D) $-\frac{3\sqrt{3}}{4} \leftarrow$

Solution. Three layers: the cube, then cosine, then $2x$. Peel them from the outside in:

$$h'(x) = 3\cos^2(2x) \cdot (-\sin(2x)) \cdot 2 = -6\cos^2(2x)\sin(2x).$$

At $x = \frac{\pi}{6}$ the inside is $2x = \frac{\pi}{3}$, where $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$:

$$h'\left(\frac{\pi}{6}\right) = -6\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{4}.$$

Answer: **(D)**. Choice (C) forgets the innermost factor 2; (A) differentiates $\cos$ as $+\sin$; (B) evaluates the trig functions at $\frac{\pi}{6}$ instead of at $2x = \frac{\pi}{3}$.**(b)** (2 points) Use the table. Let $h(x) = f(g(x))$. Circle $h'(2)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	1	4	3	-1
3	7	5	0	6

- (A) 15 (B) $-5 \leftarrow$ (C) -4 (D) 5

Solution. By the chain rule, $h'(2) = f'(g(2))g'(2)$. From the table, $g(2) = 3$, so

$$h'(2) = f'(3) \cdot g'(2) = 5 \cdot (-1) = -5.$$

Answer: **(B)**. Choice (C) evaluates f' at 2 instead of at $g(2) = 3$; (D) forgets the factor $g'(2)$; (A) multiplies by $g(2)$ instead of $g'(2)$.**(c)** (2 points) Find the derivative of $k(x) = \tan^4(2x^2 + 1)$.**Solution.** Rewrite to see the layers: $k(x) = \left(\tan(2x^2 + 1)\right)^4$. Outside to inside: the 4th power, then tangent, then $2x^2 + 1$. Peel one layer at a time:

$$\begin{aligned} k'(x) &= 4\left(\tan(2x^2 + 1)\right)^3 \cdot \frac{d}{dx}\left[\tan(2x^2 + 1)\right] && \text{power layer} \\ &= 4\tan^3(2x^2 + 1) \cdot \sec^2(2x^2 + 1) \cdot 4x && \text{tangent layer, then the inside} \\ &= 16x \tan^3(2x^2 + 1) \sec^2(2x^2 + 1). \end{aligned}$$

Grading: 1 point for the power and tangent layers (including $\frac{d}{dx} \tan u = \sec^2 u$); 1 point for the inner factor $4x$ and a correct final answer.

Problem 8. Implicit differentiation.

(6 points)

(a) (2 points) Assuming y is defined implicitly by $x^2 + 4y^2 = 20$, circle $\frac{dy}{dx}$.

- (A) $-\frac{x}{2y}$ (B) $-\frac{4y}{x}$ (C) $-\frac{x}{4y} \leftarrow$ (D) $\frac{x}{4y}$

Solution. Differentiate both sides with respect to x ; the chain rule gives $\frac{d}{dx}(4y^2) = 8y \frac{dy}{dx}$:

$$2x + 8y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{2x}{8y} = -\frac{x}{4y}.$$

Answer: **(C)**. Choice (A) differentiates $4y^2$ as $4y \frac{dy}{dx}$; (B) is the reciprocal; (D) loses the sign when moving $2x$ across.

(b) (2 points) The point $(1, 2)$ lies on the curve $x^2 + xy + y^2 = 7$. Circle the slope of the line tangent to the curve at $(1, 2)$.

- (A) $\frac{4}{5}$ (B) $-\frac{4}{5} \leftarrow$ (C) -1 (D) $-\frac{2}{5}$

Solution. Differentiate; xy needs the product rule and y^2 the chain rule:

$$2x + \left(y + x \frac{dy}{dx}\right) + 2y \frac{dy}{dx} = 0 \implies \frac{dy}{dx}(x + 2y) = -(2x + y) \implies \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

At $(1, 2)$: $\frac{dy}{dx} = -\frac{2+2}{1+4} = -\frac{4}{5}$. Answer: **(B)**.

Choice (C) differentiates xy as just y , giving $-\frac{2x+y}{2y} = -1$; (D) differentiates it as just $x \frac{dy}{dx}$, giving $-\frac{2x}{x+2y} = -\frac{2}{5}$; (A) drops the sign.

(c) (2 points) Assuming y is defined implicitly by $x^2y^3 + \cos y = 4x$, find $\frac{dy}{dx}$.

Solution. Differentiate both sides. The term x^2y^3 needs the product rule, with the chain rule on y^3 ; and $\frac{d}{dx}(\cos y) = -\sin y \frac{dy}{dx}$:

$$\underbrace{2x y^3 + x^2 \cdot 3y^2 \frac{dy}{dx}}_{\text{product rule on } x^2y^3} - \sin y \frac{dy}{dx} = 4.$$

Collect the $\frac{dy}{dx}$ terms on the left, factor, and divide:

$$\frac{dy}{dx}(3x^2y^2 - \sin y) = 4 - 2xy^3 \implies \frac{dy}{dx} = \frac{4 - 2xy^3}{3x^2y^2 - \sin y}.$$

Grading: 1 point for correctly differentiating every term (product rule, and a $\frac{dy}{dx}$ on each y -term); 1 point for solving for $\frac{dy}{dx}$.