

Integration Formulas and the Net Change Theorem

Section 5.4

Calculus I — Lecture Notes

November 10, 2026

Your speedometer never shows the trip's distance — yet the odometer does. The integral is the bridge.

We already know how to compute a definite integral using the Fundamental Theorem of Calculus: find an antiderivative, plug in the endpoints, subtract. Today we put that machinery to work. First we collect the basic **integration formulas** we will lean on all the time, and we sharpen the distinction between an *indefinite* integral (a whole family of functions) and a *definite* integral (a single number). Then comes the payoff: the **Net Change Theorem**, which says that integrating a rate of change recovers the total change in the quantity itself. That one idea powers everything from “how far did the car go” to “how much gas did the engine burn.”

Basic Integration Formulas

Every integration formula is a differentiation formula read backward. Because the derivative of x^{n+1} is $(n+1)x^n$, dividing by $n+1$ gives the single most useful rule we have.

Definition 1 (Power Rule for Integrals). For any exponent $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

The “ $+C$ ” is not optional: every function of the form $\frac{x^{n+1}}{n+1} + C$ has the same derivative x^n , so the antiderivative is really a whole family, one curve for each constant C . We also use the fact that integration is **linear** — constants pull out and sums split apart:

$$\int [a f(x) + b g(x)] dx = a \int f(x) dx + b \int g(x) dx.$$

A few trigonometric formulas round out our basic toolkit. Read each one backward as a derivative to convince yourself:

$$\int \sin x dx = -\cos x + C, \quad \int \cos x dx = \sin x + C, \quad \int \sec^2 x dx = \tan x + C.$$

Indefinite versus definite

Keep the two notions straight. An **indefinite integral** $\int f(x) dx$ represents a *family of functions* (hence the $+C$). A **definite integral** $\int_a^b f(x) dx$ is a *number* when the limits are constants — the constant C cancels in the subtraction and never appears in the answer.

Example 1 (Integrating with the power rule). Evaluate $\int_1^4 \sqrt{t}(1+t) dt$.

Solution. The power rule wants a plain power of t , so first rewrite and distribute:

$$\int_1^4 \sqrt{t}(1+t) dt = \int_1^4 t^{1/2}(1+t) dt = \int_1^4 (t^{1/2} + t^{3/2}) dt.$$

Now apply the power rule to each term:

$$\int_1^4 (t^{1/2} + t^{3/2}) dt = \left(\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} \right) \Big|_1^4.$$

Evaluate at the endpoints and subtract:

$$= \left[\frac{2}{3}(4)^{3/2} + \frac{2}{5}(4)^{5/2} \right] - \left[\frac{2}{3}(1)^{3/2} + \frac{2}{5}(1)^{5/2} \right] = \frac{256}{15}.$$

Because $4^{3/2} = 8$ and $4^{5/2} = 32$, the first bracket is $\frac{16}{3} + \frac{64}{5}$ and the second is $\frac{2}{3} + \frac{2}{5}$; combining over a common denominator gives $\frac{256}{15}$.

The Net Change Theorem

Here is the central idea of the whole section. If a quantity F changes over time, its *rate* of change is the derivative F' . The Net Change Theorem says we can run that backward: add up the rate over an interval and you recover exactly how much F changed.

Definition 2 (Net Change Theorem). The new value of a changing quantity equals its initial value plus the integral of its rate of change:

$$F(b) = F(a) + \int_a^b F'(x) dx, \quad \text{equivalently} \quad \int_a^b F'(x) dx = F(b) - F(a).$$

The second form is just the first with $F(a)$ subtracted from both sides — and it is exactly the Fundamental Theorem of Calculus. What is new is the *interpretation*: the integral of a rate is a **net change**. Net change handles negatives automatically. If F' is sometimes negative, those stretches subtract from the total, all inside a single integral, with no special bookkeeping.

Example 2 (Gasoline consumed by a motorboat). A motorboat's engine starts at $t = 0$ and burns fuel at a rate of $5 - 0.1t^3$ gallons per hour. How much gasoline is used in the first 2 hours?

Solution. The rate of consumption is a rate of change, so the total fuel used is its integral over $[0, 2]$:

$$\int_0^2 (5 - 0.1t^3) dt = \left(5t - 0.1 \frac{t^4}{4} \right) \Big|_0^2.$$

Evaluate at the endpoints:

$$= \left[5(2) - 0.1 \frac{(2)^4}{4} \right] - 0 = 10 - 0.1(4) = 10 - 0.4 = 9.6.$$

The motorboat uses 9.6 gallons of gas in the first 2 hours. Notice how the rate itself was never the answer — integrating it was.

Motion: Displacement versus Total Distance

The most important application of net change is motion. Let $v(t)$ be the velocity of an object. Velocity is the rate of change of position, so the Net Change Theorem gives two different — and easily confused — quantities.

Definition 3 (Displacement and total distance). Over the time interval $[a, b]$, the **net displacement** is

$$\int_a^b v(t) dt,$$

which counts forward motion as positive and backward motion as negative. The **total distance traveled** is

$$\int_a^b |v(t)| dt,$$

which counts every bit of motion as positive, regardless of direction.

Think of a car heading north at 40 mph for two hours, then south at 30 mph for one hour. Its *displacement* is $80 - 30 = 50$ miles (net, still north of start), but the *total distance* its odometer records is $80 + 30 = 110$ miles. Displacement is where you ended up; distance is how far you drove. The only difference in the formulas is the absolute value.

Example 3 (Net displacement of a particle). A particle moves with velocity $v(t) = 3t - 5$ (meters per second) from $t = 0$ to $t = 3$. Find the net displacement.

Solution. Displacement is the integral of velocity:

$$\int_0^3 (3t - 5) dt = \left(\frac{3t^2}{2} - 5t \right) \Big|_0^3 = \left[\frac{3(3)^2}{2} - 5(3) \right] - 0 = \frac{27}{2} - 15 = -\frac{3}{2}.$$

The net displacement is $-\frac{3}{2}$ m. The negative sign means the particle finished $\frac{3}{2}$ m *behind* where it started — early on, when $t < \frac{5}{3}$, the velocity $3t - 5$ is negative, and that backward motion outweighed the later forward motion.

Example 4 (Total distance traveled). Using the same velocity $v(t) = 3t - 5$ m/s on $[0, 3]$, find the total distance the particle travels.

Solution. Total distance integrates $|v(t)|$, so we must split the interval where the velocity changes sign. Set $v(t) = 0$:

$$3t - 5 = 0 \implies t = \frac{5}{3}.$$

On $[0, \frac{5}{3}]$ the velocity is negative, so $|v(t)| = -v(t)$; on $[\frac{5}{3}, 3]$ it is positive, so $|v(t)| = v(t)$. Therefore

$$\int_0^3 |v(t)| dt = \int_0^{5/3} (5 - 3t) dt + \int_{5/3}^3 (3t - 5) dt.$$

Antidifferentiate each piece:

$$= \left(5t - \frac{3t^2}{2} \right) \Big|_0^{5/3} + \left(\frac{3t^2}{2} - 5t \right) \Big|_{5/3}^3.$$

The first piece evaluates to $\frac{25}{3} - \frac{25}{6} = \frac{25}{6}$. The second piece evaluates to $(\frac{27}{2} - 15) - (\frac{25}{6} - \frac{50}{6}) = -\frac{3}{2} + \frac{25}{6} = \frac{16}{6}$. Adding,

$$\int_0^3 |v(t)| dt = \frac{25}{6} + \frac{16}{6} = \frac{41}{6}.$$

The total distance traveled is $\frac{41}{6}$ m. Compare this with the displacement of $-\frac{3}{2}$ m from the previous example: distance is larger because the backward and forward legs *add* instead of canceling.

Integrating Even and Odd Functions

Symmetry can save real work. Recall that f is **even** if $f(-x) = f(x)$ (graph symmetric about the y -axis) and **odd** if $f(-x) = -f(x)$ (graph symmetric about the origin). Over a symmetric interval $[-a, a]$ this symmetry collapses the integral.

Definition 4 (Integrals of Even and Odd Functions). If f is continuous and **even**, then

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx.$$

If f is continuous and **odd**, then

$$\int_{-a}^a f(x) dx = 0.$$

For an even function the area on the left of the y -axis mirrors the area on the right, so we double one half. For an odd function the signed area on the left exactly cancels the signed area on the right.

Example 5 (Integrating an odd function). Evaluate $\int_{-\pi}^{\pi} -5 \sin x dx$.

Solution. Since $\sin(-x) = -\sin x$, the function $-5 \sin x$ is odd, and we are integrating over the symmetric interval $[-\pi, \pi]$. The odd-function rule tells us the answer immediately:

$$\int_{-\pi}^{\pi} -5 \sin x dx = 0.$$

To confirm it directly, antidifferentiate:

$$\int_{-\pi}^{\pi} -5 \sin x dx = -5(-\cos x) \Big|_{-\pi}^{\pi} = 5 \cos x \Big|_{-\pi}^{\pi} = 5 \cos \pi - 5 \cos(-\pi) = -5 - (-5) = 0.$$

The positive area over $[-\pi, 0]$ is exactly canceled by the negative area over $[0, \pi]$.

Practice (work this out). A particle moves along a straight line with velocity $v(t) = 4 - 2t$ (meters per second) for $0 \leq t \leq 3$. Find both the net displacement and the total distance traveled up to $t = 3$.

Practice (work this out). Evaluate $\int_0^{\pi} (\sin x - \cos x) dx$.

Summary

- **Power rule for integrals:** $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$; integration is linear (pull out constants, split sums).
- **Trig integrals:** $\int \sin x dx = -\cos x + C$, $\int \cos x dx = \sin x + C$, $\int \sec^2 x dx = \tan x + C$.
- An **indefinite** integral is a family of functions (keep the $+C$); a **definite** integral with constant limits is a single number.
- **Net Change Theorem:** the integral of a rate of change equals the total change, $\int_a^b F'(x) dx = F(b) - F(a)$; it accounts for negative contributions automatically.
- **Motion:** net displacement is $\int_a^b v(t) dt$; total distance is $\int_a^b |v(t)| dt$ — split the interval where v changes sign.
- **Symmetry shortcuts:** over $[-a, a]$, an even function gives $2 \int_0^a f$, an odd function gives 0.
- Next time: integration by substitution (Section 5.5), the chain rule run backward.