

Integration by Substitution

Section 5.5

Calculus I — Lecture Notes

November 12, 2026

The chain rule made derivatives easy; run it backward and it computes integrals.

The Fundamental Theorem of Calculus told us that to evaluate an integral we only need an *antiderivative* — no Riemann sums required. The catch is that finding an antiderivative is not always easy. So far our only integrals have been the ones we could guess directly. Today we get our first real technique for building antiderivatives: **integration by substitution**, also called **change of variables**. The idea is simple to state — it is the chain rule read from right to left — and once you can spot the pattern, a huge family of integrals opens up.

Reversing the Chain Rule

Recall the chain rule. If we differentiate $F(g(x))$, we get

$$\frac{d}{dx} F(g(x)) = F'(g(x)) g'(x).$$

Now integrate both sides. The left side just undoes the derivative, so

$$\int F'(g(x)) g'(x) dx = F(g(x)) + C.$$

Write $f = F'$. Then the integrand has the shape $f(g(x)) g'(x)$: an *inside function* $g(x)$, its derivative $g'(x)$ sitting alongside, and f wrapped around the outside. Whenever an integrand looks like this, we can collapse the inside into a single new variable u .

Definition 1 (Substitution with indefinite integrals). Let $u = g(x)$, where g' is continuous, and let f be continuous on the range of g . If F is an antiderivative of f , then

$$\int f(g(x)) g'(x) dx = \int f(u) du = F(u) + C = F(g(x)) + C.$$

The mechanics are bookkeeping. We set $u = g(x)$, compute the differential $du = g'(x) dx$, and swap both pieces into the integral. Done well, every trace of x disappears and we are left with a clean integral in u that we already know how to do. At the end we put x back.

Strategy. Pick $u = g(x)$ so that $g'(x)$ is (up to a constant) also in the integrand. Replace $g(x)$ by u and $g'(x) dx$ by du . Integrate in u , then substitute back.

The Basic Pattern

Example 1 (The derivative is already there). Find $\int 6x(3x^2 + 4)^4 dx$.

Solution. The messy part is the inside $3x^2 + 4$, so let

$$u = 3x^2 + 4, \quad du = 6x dx.$$

Look at the integrand: the factor $6x dx$ is *exactly* du . Substitute:

$$\int 6x(3x^2 + 4)^4 dx = \int u^4 du.$$

Now use the power rule for integrals:

$$\int u^4 du = \frac{u^5}{5} + C.$$

Finally, replace u with $3x^2 + 4$:

$$\int 6x(3x^2 + 4)^4 dx = \frac{(3x^2 + 4)^5}{5} + C.$$

Check by differentiating: $\frac{d}{dx} \frac{1}{5}(3x^2 + 4)^5 = \frac{1}{5} \cdot 5(3x^2 + 4)^4 \cdot 6x = 6x(3x^2 + 4)^4$. That is the original integrand, so the answer is correct.

The whole method rides on one habit: *differentiate your guess to check it*. If the derivative reproduces the integrand, you are done, no matter how you found the antiderivative.

Adjusting a Constant

Often the derivative of the inside function is present up to a numerical factor. That is fine — a constant can be moved across the integral sign freely, so we patch the mismatch by hand.

Example 2 (Fixing a constant factor). Find $\int z\sqrt{z^2 - 5} dz$.

Solution. Rewrite the root as a power and choose the inside as u :

$$\int z(z^2 - 5)^{1/2} dz, \quad u = z^2 - 5, \quad du = 2z dz.$$

The integrand has only $z dz$, but $du = 2z dz$ — twice as much as we have. Solve for what we need by multiplying the du equation by $\frac{1}{2}$:

$$\frac{1}{2} du = z dz.$$

Now substitute, pulling the constant $\frac{1}{2}$ outside:

$$\int z(z^2 - 5)^{1/2} dz = \int u^{1/2} \cdot \frac{1}{2} du = \frac{1}{2} \int u^{1/2} du.$$

Integrate in u :

$$\frac{1}{2} \int u^{1/2} du = \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{3} u^{3/2} + C.$$

Put z back:

$$\int z\sqrt{z^2 - 5} dz = \frac{1}{3} (z^2 - 5)^{3/2} + C.$$

A word of caution: you may only move *constants* across the integral sign this way. If the leftover factor were a variable that is not part of du , this trick alone would not work.

A Trigonometric Integrand

Substitution shines on trig integrals, because the derivatives of sine and cosine feed each other. Remember $\frac{d}{dx} \cos x = -\sin x$ and $\frac{d}{dx} \sin x = \cos x$; one supplies the du for the other.

Example 3 (Substitution with a trig function). Evaluate $\int \frac{\sin t}{\cos^3 t} dt$.

Solution. The denominator is a power of $\cos t$, whose derivative is $-\sin t$ — and a $\sin t$ is sitting in the numerator. So let

$$u = \cos t, \quad du = -\sin t \, dt.$$

Then $\sin t \, dt = -du$. Substituting,

$$\int \frac{\sin t}{\cos^3 t} dt = \int \frac{-du}{u^3} = -\int u^{-3} du.$$

Integrate with the power rule:

$$-\int u^{-3} du = -\frac{u^{-2}}{-2} + C = \frac{1}{2} u^{-2} + C = \frac{1}{2u^2} + C.$$

Return to t with $u = \cos t$:

$$\int \frac{\sin t}{\cos^3 t} dt = \frac{1}{2 \cos^2 t} + C.$$

Practice (work this out). Use substitution to find $\int \cos^3 t \sin t \, dt$. (*Hint: let $u = \cos t$.*)

Substitution for Definite Integrals

Substitution works for definite integrals too, with one extra step. The limits of integration are values of x ; once we change variables to u , those limits must become values of u . If $u = g(x)$, then as x runs from a to b , the new variable runs from $g(a)$ to $g(b)$.

Definition 2 (Substitution with definite integrals). Let $u = g(x)$ with g' continuous on $[a, b]$, and let f be continuous on the range of g . Then

$$\int_a^b f(g(x)) g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.$$

The payoff: once you convert the limits, you never have to switch back to x . You evaluate the u -integral straight from $g(a)$ to $g(b)$.

Example 4 (Changing the limits). Evaluate $\int_0^1 x^2 (1 + 2x^3)^5 \, dx$.

Solution. Let the inside be u :

$$u = 1 + 2x^3, \quad du = 6x^2 \, dx.$$

The integrand has $x^2 \, dx$, so divide the du equation by 6:

$$\frac{1}{6} du = x^2 \, dx.$$

Now convert the limits using $u = 1 + 2x^3$:

$$x = 0 \Rightarrow u = 1 + 2(0)^3 = 1, \quad x = 1 \Rightarrow u = 1 + 2(1)^3 = 3.$$

Substitute everything — integrand, differential, and limits:

$$\int_0^1 x^2 (1 + 2x^3)^5 dx = \frac{1}{6} \int_1^3 u^5 du.$$

Evaluate:

$$\frac{1}{6} \int_1^3 u^5 du = \frac{1}{6} \cdot \frac{u^6}{6} \Big|_1^3 = \frac{1}{36} [3^6 - 1^6] = \frac{729 - 1}{36} = \frac{728}{36} = \frac{182}{9}.$$

Example 5 (A trig identity, then substitution). Evaluate $\int_0^{\pi/2} \cos^2 \theta d\theta$.

Solution. There is no obvious inside function, so first rewrite the integrand with the power-reduction identity $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$:

$$\int_0^{\pi/2} \cos^2 \theta d\theta = \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{2} \int_0^{\pi/2} 1 d\theta + \frac{1}{2} \int_0^{\pi/2} \cos 2\theta d\theta.$$

The first integral is immediate:

$$\frac{1}{2} \int_0^{\pi/2} 1 d\theta = \frac{1}{2} \cdot [\theta]_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

For the second, substitute $u = 2\theta$, so $du = 2 d\theta$ and $\frac{1}{2} du = d\theta$. The limits change: $\theta = 0 \Rightarrow u = 0$ and $\theta = \frac{\pi}{2} \Rightarrow u = \pi$. Thus

$$\frac{1}{2} \int_0^{\pi/2} \cos 2\theta d\theta = \frac{1}{2} \cdot \frac{1}{2} \int_0^{\pi} \cos u du = \frac{1}{4} [\sin u]_0^{\pi} = \frac{1}{4} (\sin \pi - \sin 0) = \frac{1}{4} (0 - 0) = 0.$$

Adding the two pieces,

$$\int_0^{\pi/2} \cos^2 \theta d\theta = \frac{\pi}{4} + 0 = \frac{\pi}{4}.$$

Practice (work this out). Use a change of variables to evaluate $\int_0^1 x\sqrt{1-x^2} dx$. (*Hint: let $u = 1 - x^2$ and change the limits.*)

Summary

- **Substitution reverses the chain rule.** It applies when the integrand has the form $f(g(x)) g'(x)$: an inside function and (up to a constant) its derivative.
- **The procedure:** choose $u = g(x)$, compute $du = g'(x) dx$, replace both pieces so only u remains, integrate, then substitute back.
- **Adjust constants freely.** If du differs from what the integrand supplies by a numerical factor, multiply or divide to match — but only constants may cross the integral sign.
- **Trig integrands** are natural targets, since sin and cos supply each other's derivatives; a trig identity sometimes has to come first.

- **For definite integrals, change the limits.** Convert $x = a, b$ into $u = g(a), g(b)$ and evaluate in u — no need to return to x .
- **Always check** by differentiating your answer: it should reproduce the original integrand.
- Next time: Midterm 2 (Chapters 4–5) on November 17; then Section 6.1, areas between curves, on November 19.