

The Definite Integral

Section 5.2

Calculus I — Lecture Notes

November 3, 2026

From Riemann Sums to a Single Number

Add up infinitely many infinitely thin slices, and out pops one clean number.

Last time we chopped the region under a curve into thin rectangles, added up their areas, and took the limit as the rectangles got skinnier. That gave us the **area under a curve**:

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

But that picture came with fine print: we required $f(x)$ to be continuous *and* nonnegative. Real problems don't always cooperate — functions dip below the axis, velocities point backward. Today we keep the same limit-of-sums machinery but throw away those restrictions. The result is the **definite integral**: a single number attached to a function and an interval, which we will learn to read as *signed area*.

Definition and Notation

Definition 1. If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of f from a to b is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x,$$

provided the limit exists. If it does, we say f is **integrable** on $[a, b]$.

This symbol should look familiar — but read the notation carefully, because a *definite* integral is a **number**, not a family of functions. The elongated “S,” written $\int$, was Leibniz's shorthand for a sum. The numbers a and b sitting below and above it are the **limits of integration**: a is the lower limit, b the upper limit. (Watch out — the word “limit” is doing double duty here: the limit of the sum as $n \rightarrow \infty$, and the endpoints of the interval.) The function $f(x)$ inside is the **integrand**, and the dx marks x as the **variable of integration**.

That variable is a *dummy*: it never survives to the answer, so its name is irrelevant. All three of these are the same number:

$$\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du.$$

When is f actually integrable? Continuity is enough.

Definition 2 (Continuous functions are integrable). If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Continuity is sufficient but not necessary: a function with only a few jump discontinuities on $[a, b]$ can still be integrable. We also keep using *regular* partitions — equal-width rectangles, $\Delta x = (b - a)/n$ — because they make the arithmetic clean without losing any real insight.

Example 1 (Evaluating an integral from the definition). Use the definition to evaluate $\int_0^2 x^2 dx$ with a right-endpoint Riemann sum.

Solution. Here $a = 0$ and $b = 2$, so the width of each rectangle is

$$\Delta x = \frac{b - a}{n} = \frac{2}{n}.$$

Using the right endpoint of the i -th subinterval,

$$x_i = 0 + i \Delta x = \frac{2i}{n}, \quad f(x_i) = x_i^2 = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}.$$

The Riemann sum becomes

$$\sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \frac{4i^2}{n^2} \cdot \frac{2}{n} = \frac{8}{n^3} \sum_{i=1}^n i^2.$$

Now use the summation formula $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$:

$$\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{8}{n^3} \cdot \frac{2n^3 + 3n^2 + n}{6} = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}.$$

Finally take the limit as $n \rightarrow \infty$; the last two terms vanish:

$$\int_0^2 x^2 dx = \lim_{n \rightarrow \infty} \left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} \right) = \frac{8}{3}.$$

Doing every integral this way is exhausting. The point of the definition is to know *what the number means*. Very soon we will find shortcuts — and the first shortcut is geometry.

Area and the Definite Integral

Since the definite integral is built from areas of rectangles, whenever the region has a shape we already know — a triangle, a rectangle, a circle — we can read the integral straight off a formula, no limits required.

Example 2 (A definite integral from a geometric formula). Use the area of a circle to evaluate

$$\int_3^6 \sqrt{9 - (x - 3)^2} dx.$$

Solution. Set $y = \sqrt{9 - (x - 3)^2}$. Squaring gives $(x - 3)^2 + y^2 = 9$ with $y \geq 0$: the top half of a circle of radius 3 centered at $(3, 0)$. Over $[3, 6]$ we are looking at the right half of that semicircle — one *quarter* of the full circle. A full circle of radius r has area πr^2 , so

$$\int_3^6 \sqrt{9 - (x - 3)^2} dx = \frac{1}{4} \pi r^2 = \frac{1}{4} \pi (3)^2 = \frac{9\pi}{4} \approx 7.069.$$

Net signed area

We dropped the requirement that $f \geq 0$, so what does “area” mean when the curve dips below the axis? Go back to the Riemann sum. When $f(x_i^*) > 0$, the term $f(x_i^*) \Delta x$ is the area of a rectangle above the axis. When $f(x_i^*) < 0$, that same term is *negative* — it counts as the negative of the rectangle’s area. So in the limit,

$$\int_a^b f(x) dx = A_1 - A_2,$$

where A_1 is the area *above* the x -axis and A_2 is the area *below* it. This quantity is the **net signed area**. It can be positive, negative, or zero: positive when more area sits above the axis, negative when more sits below, and zero when they balance.

If instead we want the **total area** — how much region there is, regardless of side — we add the pieces rather than subtract, which we arrange with an absolute value:

$$\int_a^b |f(x)| dx = A_1 + A_2.$$

Example 3 (Net signed area). Find the net signed area between $f(x) = 2x$ and the x -axis over $[-3, 3]$.

Solution. The line $y = 2x$ forms two triangles: one below the axis from $x = -3$ to $x = 0$, and one above it from $x = 0$ to $x = 3$. Each has base 3. At $x = 3$ the height is $2(3) = 6$, and at $x = -3$ the depth is likewise 6. Using $A = \frac{1}{2}bh$:

$$A_1 = \frac{1}{2}(3)(6) = 9 \quad (\text{above the axis}),$$

$$A_2 = \frac{1}{2}(3)(6) = 9 \quad (\text{below the axis}).$$

The net signed area is

$$\int_{-3}^3 2x dx = A_1 - A_2 = 9 - 9 = 0.$$

The two triangles are mirror images, so they cancel exactly.

Example 4 (Total area). Find the total area between $f(x) = x - 2$ and the x -axis over $[0, 6]$.

Solution. The line crosses the axis at $x = 2$ (set $y = 0$). On $[0, 2]$ it lies *below* the axis; on $[2, 6]$ it lies *above*. For total area we add both pieces:

$$\int_0^6 |x - 2| dx = A_2 + A_1.$$

The lower triangle on $[0, 2]$ has base 2 and, at $x = 0$, depth $|0 - 2| = 2$:

$$A_2 = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2.$$

The upper triangle on $[2, 6]$ has base 4 and, at $x = 6$, height $6 - 2 = 4$:

$$A_1 = \frac{1}{2}bh = \frac{1}{2}(4)(4) = 8.$$

The total area is

$$A_1 + A_2 = 8 + 2 = 10.$$

Note this is *not* the same as the net signed area $8 - 2 = 6$: total area never lets the pieces cancel.

Practice (work this out). Use geometry (area formulas) to evaluate $\int_2^4 (2x+3) dx$ as a net signed area. (Sketch the region first — it is a trapezoid, which splits into a rectangle and a triangle.)

Properties of the Definite Integral

Once we think of the integral as signed area, a handful of properties become almost obvious. They let us break a hard integral into easy pieces.

Definition 3 (Properties of the definite integral). Let f and g be integrable, and let c be a constant.

1. $\int_a^a f(x) dx = 0$ (no width, no area).
2. $\int_b^a f(x) dx = -\int_a^b f(x) dx$ (reversing the limits flips the sign).
3. $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$.
4. $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$.
5. $\int_a^b c f(x) dx = c \int_a^b f(x) dx$ (constants pull out front).
6. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ (split the interval at any point c).

Property 6 holds for *any* c , not just one between a and b , as long as f is integrable across the whole span. Properties 3, 4, and 5 together say integration is **linear**: it respects sums, differences, and constant multiples.

Example 5 (Using the properties). Suppose $\int_0^8 f(x) dx = 10$ and $\int_0^5 f(x) dx = 5$. Find $\int_5^8 f(x) dx$.

Solution. Split the interval $[0, 8]$ at $c = 5$ using property 6:

$$\int_0^8 f(x) dx = \int_0^5 f(x) dx + \int_5^8 f(x) dx.$$

Substitute the known values and solve:

$$10 = 5 + \int_5^8 f(x) dx \quad \implies \quad \int_5^8 f(x) dx = 5.$$

Example 6 (Comparing two integrals). Suppose $f(x) \geq g(x)$ for every x in $[a, b]$. Which is larger, $\int_a^b f(x) dx$ or $\int_a^b g(x) dx$?

Solution. If f sits at or above g across the whole interval, then at every input the integrand of the first integral is at least as big as the integrand of the second. Adding up rectangles that are each at least as tall gives at least as much signed area, so

$$\int_a^b f(x) dx \geq \int_a^b g(x) dx.$$

A larger integrand over the same interval never produces a smaller integral. This is the **comparison** property, and it is why bounding a function above and below lets us trap its integral between two numbers.

The Average Value of a Function

Averaging a *list* of numbers is easy: add them, divide by how many there are. But how do you average a function, which takes on infinitely many values across an interval? Sample it.

Divide $[a, b]$ into n equal subintervals of width $\Delta x = (b - a)/n$, pick a point x_i^* in each, and average those samples:

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}.$$

Since $\Delta x = \frac{b-a}{n}$, we have $\frac{1}{n} = \frac{\Delta x}{b-a}$, so this rearranges into

$$\frac{1}{b-a} \sum_{i=1}^n f(x_i^*) \Delta x.$$

Inside is a Riemann sum. Taking the limit as $n \rightarrow \infty$ turns the sum into an integral, and the approximation becomes exact.

Definition 4. Let f be continuous on $[a, b]$. The **average value** of f on $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Geometrically: f_{ave} is the height of the rectangle over $[a, b]$ whose area equals the signed area under f .

Example 7 (Average value of a linear function). Find the average value of $f(x) = x + 1$ over $[0, 5]$.

Solution. First evaluate the integral. Over $[0, 5]$ the region under $y = x + 1$ is a trapezoid lying on its side, with height (horizontal span) 5 and parallel sides of length $f(0) = 1$ and $f(5) = 6$. Using $A = \frac{1}{2}h(a + b)$:

$$\int_0^5 (x + 1) dx = \frac{1}{2}(5)(1 + 6) = \frac{35}{2}.$$

Now divide by the width of the interval:

$$f_{\text{ave}} = \frac{1}{5-0} \int_0^5 (x + 1) dx = \frac{1}{5} \cdot \frac{35}{2} = \frac{7}{2}.$$

So on average, $f(x) = x + 1$ takes the value $\frac{7}{2}$ over $[0, 5]$ — exactly its value at the midpoint $x = \frac{5}{2}$, as you would expect for a straight line.

Practice (work this out). Find the average value of $f(x) = 6 - 2x$ over $[0, 3]$. (Hint: the region is a triangle with base 3 and height 6.)

Summary

- The **definite integral** $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ is a single *number*, the limit of Riemann sums; it exists whenever f is continuous on $[a, b]$.
- Notation: a, b are the **limits of integration**, $f(x)$ is the **integrand**, and the variable of integration is a dummy ($\int f(x) dx = \int f(t) dt$).

- The integral measures **net signed area**: area above the axis minus area below, $A_1 - A_2$. **Total area** adds them, $\int_a^b |f| dx = A_1 + A_2$.
- Known shapes (triangles, rectangles, circles) let you evaluate integrals by geometry instead of limits.
- **Properties**: equal limits give 0; reversing limits flips the sign; integration is linear (sums, differences, constants); you may split the interval at any point.
- The **average value** of f on $[a, b]$ is $f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx$.
- Next time: the Fundamental Theorem of Calculus, which finally frees us from computing these limits by hand.