

The Fundamental Theorem of Calculus

Section 5.3

Calculus I — Lecture Notes

November 5, 2026

The derivative and the integral are two sides of the same coin — and today we get to see why.

The Payoff

All semester we have chased two questions. **How fast is it changing?** (the derivative.) **How much has piled up?** (the integral.) So far these have lived in separate worlds: derivatives came from limits of slopes, integrals came from limits of Riemann sums. Computing a definite integral meant either a lucky area formula or a brutal limit of sums — both painful.

Today those two worlds collide. The **Fundamental Theorem of Calculus** says that differentiation and integration are *inverse operations*. One undoes the other. This single fact turns the impossible-looking problem “add up infinitely many infinitely thin slices” into the easy problem “find an antiderivative and subtract.” This is the result the whole course has been pointing at. Newton and Leibniz found it in the late 1600s, and it changed what humans could compute. Before we get there, we need one supporting result.

The Mean Value Theorem for Integrals

A continuous function on a closed interval has an average value (from the last section, $\frac{1}{b-a} \int_a^b f(x) dx$). The next theorem says the function actually *hits* that average somewhere on the interval — it does not just hover around it.

Definition 1 (Mean Value Theorem for Integrals). If $f(x)$ is continuous over $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

Equivalently, $\int_a^b f(x) dx = f(c) (b-a)$.

The second form has a clean picture: the area under the curve equals the area of a single rectangle of width $b-a$ whose height $f(c)$ is a value the function truly takes. Why must such a c exist? Since f is continuous on $[a, b]$ it has a minimum m and a maximum M , so $m \leq f(x) \leq M$ everywhere, and the average value is squeezed between m and M . A continuous function that reaches m and M must also reach every value in between — so it reaches the average.

Example 1 (Finding where a function meets its average). Find the average value of $f(x) = 8 - 2x$ over $[0, 4]$, and find c so that $f(c)$ equals that average.

Solution. The average value is

$$\frac{1}{4-0} \int_0^4 (8-2x) dx.$$

The graph of $y = 8 - 2x$ over $[0, 4]$ is a straight line forming a right triangle with the axes, with base 4 and height 8, so

$$\int_0^4 (8 - 2x) dx = \frac{1}{2}(4)(8) = 16.$$

The average value is $\frac{1}{4}(16) = 4$. Now set $f(c) = 4$ and solve:

$$8 - 2c = 4 \implies c = 2.$$

So at $c = 2$ the function equals its own average, $f(2) = 4$. The theorem promised such a point, and here it is.

Practice (work this out). Find the average value of $f(x) = \frac{x}{2}$ over $[0, 6]$, and find c in $[0, 6]$ such that $f(c)$ equals that average value.

FTC Part 1: The Derivative of an Integral

Here is the first half of the payoff. Fix a continuous function f and a starting point a . Define a new function by letting the upper limit of an integral vary:

$$F(x) = \int_a^x f(t) dt.$$

Read this carefully. For each particular x , the right side is a *number* (a definite integral, the accumulated area from a out to x). As x moves, that number changes — so F is a genuine function of x . We use t as the integration variable just to keep it from clashing with the limit x .

Definition 2 (Fundamental Theorem of Calculus, Part 1). If $f(x)$ is continuous over $[a, b]$ and $F(x) = \int_a^x f(t) dt$, then

$$F'(x) = f(x) \quad \text{on } (a, b).$$

In words: *the derivative of an accumulated area is the function you were accumulating.* Integration builds F up; differentiation tears it back down to f . They undo each other. As a bonus, this guarantees every continuous function has an antiderivative — namely its own accumulation function.

Why is it true? By the definition of the derivative,

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt.$$

That last integral divided by h is exactly the average value of f over $[x, x+h]$. By the Mean Value Theorem for Integrals it equals $f(c)$ for some c between x and $x+h$. As $h \rightarrow 0$, that c is trapped against x , and since f is continuous, $f(c) \rightarrow f(x)$. So $F'(x) = f(x)$. (Notice we needed the theorem from the previous section to prove this one.)

Example 2 (Differentiating an integral). Use FTC Part 1 to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3 + 1} dt.$$

Solution. No antiderivative-hunting needed. The theorem says the derivative is just the integrand with t replaced by x :

$$g'(x) = \frac{1}{x^3 + 1}.$$

That is the whole point of Part 1 — differentiating cancels the integrating.

When the upper limit is not simply x but a function of x , we combine Part 1 with the **chain rule**.

Example 3 (Two variable limits of integration). Let $F(x) = \int_x^{2x} t^3 dt$. Find $F'(x)$.

Solution. FTC Part 1 needs the lower limit constant, so split at 0 and flip the first piece:

$$F(x) = \int_x^{2x} t^3 dt = \int_x^0 t^3 dt + \int_0^{2x} t^3 dt = -\int_0^x t^3 dt + \int_0^{2x} t^3 dt.$$

Differentiate each piece. The first is a plain Part-1 derivative:

$$\frac{d}{dx} \left[-\int_0^x t^3 dt \right] = -x^3.$$

For the second, let $u = 2x$ and apply the chain rule: the derivative is the integrand at u times $\frac{du}{dx}$,

$$\frac{d}{dx} \left[\int_0^{2x} t^3 dt \right] = (2x)^3 \cdot 2 = 16x^3.$$

Adding the two pieces,

$$F'(x) = -x^3 + 16x^3 = 15x^3.$$

Practice (work this out). Use FTC Part 1 to find $F'(x)$ for $F(x) = \int_1^{x^3} \cos t dt$.

FTC Part 2: Evaluating Definite Integrals

Part 1 tells us differentiation undoes integration. Part 2 runs the machine the other way and hands us a practical tool: to compute a definite integral, find *any* antiderivative and subtract its values at the endpoints. No Riemann sums, no area formulas.

Definition 3. A function F is an **antiderivative** of f if $F'(x) = f(x)$. For example, $\frac{x^3}{3}$ is an antiderivative of x^2 , and so is $\frac{x^3}{3} + 7$. Running the power rule backward gives one for every power except x^{-1} :

$$\text{an antiderivative of } x^n \text{ is } \frac{x^{n+1}}{n+1} \quad (n \neq -1).$$

Check any antiderivative by differentiating it: you should get f back.

Definition 4 (Fundamental Theorem of Calculus, Part 2). If f is continuous over $[a, b]$ and $F(x)$ is *any* antiderivative of $f(x)$ (so $F' = f$), then

$$\int_a^b f(x) dx = F(b) - F(a).$$

The shorthand for $F(b) - F(a)$ is $F(x) \Big|_a^b$: evaluate the antiderivative at the top limit, subtract its value at the bottom limit. It almost feels like cheating that the area under a whole curved region collapses to two point evaluations.

Why does *any* antiderivative work, with no “+ C ”? If F and G are both antiderivatives of f , they differ by a constant, and that constant cancels in $F(b) - F(a)$. So we always choose $C = 0$.

Example 4 (A basic evaluation). Use FTC Part 2 to evaluate $\int_{-2}^2 (t^2 - 4) dt$.

Solution. By the power rule, an antiderivative of $t^2 - 4$ is $\frac{t^3}{3} - 4t$. Evaluate at the endpoints and subtract:

$$\int_{-2}^2 (t^2 - 4) dt = \left. \frac{t^3}{3} - 4t \right|_{-2}^2 = \left[\frac{(2)^3}{3} - 4(2) \right] - \left[\frac{(-2)^3}{3} - 4(-2) \right].$$

Compute each bracket:

$$= \left(\frac{8}{3} - 8 \right) - \left(-\frac{8}{3} + 8 \right) = \frac{8}{3} - 8 + \frac{8}{3} - 8 = \frac{16}{3} - 16 = -\frac{32}{3}.$$

The answer is negative because the region sits below the x -axis. Area is always positive, but a definite integral records *net signed area*.

Example 5 (Rewrite before you integrate). Evaluate $\int_1^9 \frac{x-1}{\sqrt{x}} dx$.

Solution. There is no power rule for a quotient, so first rewrite the integrand. Using $\sqrt{x} = x^{1/2}$ and splitting the fraction,

$$\frac{x-1}{x^{1/2}} = \frac{x}{x^{1/2}} - \frac{1}{x^{1/2}} = x^{1/2} - x^{-1/2}.$$

Now the power rule applies to each term:

$$\int_1^9 (x^{1/2} - x^{-1/2}) dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} \right) \Big|_1^9 = \left(\frac{2}{3} x^{3/2} - 2x^{1/2} \right) \Big|_1^9.$$

At $x = 9$: $\frac{2}{3}(27) - 2(3) = 18 - 6 = 12$. At $x = 1$: $\frac{2}{3}(1) - 2(1) = \frac{2}{3} - 2 = -\frac{4}{3}$. Subtract:

$$\int_1^9 \frac{x-1}{\sqrt{x}} dx = 12 - \left(-\frac{4}{3} \right) = \frac{40}{3}.$$

The lesson: massage the integrand into powers of x first, then evaluate.

Practice (work this out). Use FTC Part 2 to evaluate $\int_1^2 x^{-4} dx$.

How the Two Halves Connect

The two parts are mirror images of one statement: *differentiation and integration are inverse operations*.

- **Part 1** starts with f , integrates it into $F(x) = \int_a^x f$, and differentiating gives f back: $\frac{d}{dx} \int_a^x f(t) dt = f(x)$.
- **Part 2** starts with an antiderivative F (so $F' = f$) and integrates f to recover the change in F : $\int_a^b f(x) dx = F(b) - F(a)$.

Go around the loop either way and you come back to where you started. That is the sense in which the derivative and the integral are inverses — the resolution of the two questions we opened the course with. It is also why Part 2 is a computational gift: rate-of-change information (f) is turned into total accumulation ($F(b) - F(a)$) with a single subtraction.

Return to the falling ball from day one. Gravity gives the acceleration; the integral of acceleration is velocity, and the integral of velocity is displacement. The Fundamental Theorem is exactly the bridge that let us walk from “how it changes” to “how far it went” — and now we know precisely why that bridge holds.

Summary

- **Mean Value Theorem for Integrals:** a continuous f on $[a, b]$ actually attains its average value at some $c \in [a, b]$, so $\int_a^b f = f(c)(b - a)$.
- **FTC Part 1:** if $F(x) = \int_a^x f(t) dt$, then $F'(x) = f(x)$. Differentiating an accumulated integral gives back the integrand.
- With a variable upper limit $u(x)$, use the **chain rule**: $\frac{d}{dx} \int_a^{u(x)} f(t) dt = f(u(x)) u'(x)$; split at a constant when both limits vary.
- **FTC Part 2:** if $F' = f$, then $\int_a^b f(x) dx = F(b) - F(a)$. Any antiderivative works, so drop the $+C$; net signed area can be negative.
- **The big picture:** differentiation and integration are inverse operations — the two halves of calculus, finally joined. This is the result the whole course was built toward.
- Next time: integration formulas and the Net Change Theorem (Section 5.4).