

Related Rates and Linear Approximations

Sections 4.1 & 4.2

Calculus I — Lecture Notes

October 1, 2026

If you know how fast one thing is changing, the derivative tells you how fast everything tied to it is changing too.

Up to now we have taken a formula and asked for its rate of change. Today we turn that around. In the real world, several quantities are usually changing *at once*, and they are linked by an equation. Air pumps into a balloon, so both its radius and its volume grow; a ladder slides down a wall, so its foot slides out along the floor. Because the quantities are related, so are their rates. Differentiating the equation that ties them together lets us solve for the rate we want from the rates we know. In the second half we use the same derivative in a different way: the tangent line hugs a curve so closely near a point that we can use it to *estimate* values — like $\sqrt{101}$ — in our heads, and to measure how a small error in an input spreads to the output.

Setting Up a Related-Rates Problem

Suppose two quantities are both functions of time t and are connected by an equation. If we differentiate that equation with respect to t — using the chain rule, since each quantity depends on t — we get a new equation relating their **rates of change**. That is the whole idea.

Definition 1. Two quantities whose rates of change are linked through an equation are called **related rates**. If V and r are related by an equation, then $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are related by the equation obtained from differentiating both sides with respect to t .

A reliable strategy, which we follow in every example below:

1. Assign symbols to every changing quantity, and draw a figure.
2. Write down the rates you are *given* and the rate you must *find*.
3. Find an equation relating the quantities from Step 1.
4. Differentiate both sides with respect to t (chain rule!). This relates the rates.
5. Only now substitute the known values, and solve for the unknown rate.

The one trap to avoid: **do not plug in numbers too early**. If you substitute a value for a *changing* quantity before you differentiate, that quantity looks constant and its derivative vanishes — and your answer is wrong. Substitute in Step 5, never before.

Example 1 (Inflating a balloon). A spherical balloon is filled with air at the constant rate of $2 \text{ cm}^3/\text{sec}$. How fast is the radius increasing when the radius is 3 cm?

Solution. Let r be the radius and V the volume, both functions of t . We are given $\frac{dV}{dt} = 2 \text{ cm}^3/\text{sec}$ and want $\frac{dr}{dt}$ when $r = 3$. The volume of a sphere relates them:

$$V = \frac{4}{3}\pi r^3.$$

Differentiate both sides with respect to t , applying the chain rule to r^3 :

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Now substitute $\frac{dV}{dt} = 2$ and $r = 3$:

$$2 = 4\pi(3)^2 \frac{dr}{dt} = 36\pi \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{2}{36\pi} = \frac{1}{18\pi} \text{ cm/sec.}$$

So the radius grows at $\frac{1}{18\pi} \approx 0.0177$ cm/sec at that instant — slow, because a fixed trickle of air spreads over a large surface.

Example 2 (A sliding ladder). A 10-ft ladder leans against a wall. The bottom is pulled away from the wall at 2 ft/sec. How fast is the top sliding down when the bottom is 6 ft from the wall?

Solution. Let x be the distance from the wall to the foot of the ladder and y the height of the top on the wall. The ladder, wall, and floor form a right triangle with hypotenuse 10, so

$$x^2 + y^2 = 10^2 = 100.$$

We are given $\frac{dx}{dt} = 2$ ft/sec and want $\frac{dy}{dt}$ when $x = 6$. Differentiate both sides with respect to t :

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \implies x \frac{dx}{dt} + y \frac{dy}{dt} = 0.$$

When $x = 6$, the Pythagorean relation gives $y = \sqrt{100 - 36} = 8$. Substitute $x = 6$, $y = 8$, $\frac{dx}{dt} = 2$:

$$6(2) + 8 \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{12}{8} = -\frac{3}{2} \text{ ft/sec.}$$

The top slides down at $\frac{3}{2}$ ft/sec. The sign is negative because the height is *decreasing*, exactly as it should be.

Example 3 (Water draining from a cone). Water drains from a cone-shaped funnel at 0.03 ft³/sec. The funnel is 2 ft tall with a top radius of 1 ft. How fast is the water level dropping when the water is $\frac{1}{2}$ ft deep?

Solution. Let h be the depth of the water, r the radius of the water's surface, and V its volume. We are given $\frac{dV}{dt} = -0.03$ ft³/sec (negative: water is leaving) and want $\frac{dh}{dt}$ when $h = \frac{1}{2}$. The volume of a cone is

$$V = \frac{1}{3}\pi r^2 h.$$

This has *two* changing lengths, but the water cone is similar to the funnel, so its proportions match: $\frac{r}{h} = \frac{1}{2}$, i.e. $r = \frac{h}{2}$. Substituting removes r :

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$$

Differentiate with respect to t :

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

Substitute $\frac{dV}{dt} = -0.03$ and $h = \frac{1}{2}$:

$$-0.03 = \frac{\pi}{4} \left(\frac{1}{2}\right)^2 \frac{dh}{dt} = \frac{\pi}{16} \frac{dh}{dt} \implies \frac{dh}{dt} = -\frac{0.48}{\pi} \approx -0.153 \text{ ft/sec.}$$

The level falls at about 0.153 ft/sec. Reducing to the single variable h *before* differentiating is what made this manageable.

Practice (work this out). A 25-ft ladder leans against a wall and its foot is pulled away at 3 ft/sec. How fast is the top sliding down when the foot is 7 ft from the wall? (First find the height, then differentiate $x^2 + y^2 = 625$.)

Linear Approximation at a Point

Now the derivative earns its keep a second way. If f is differentiable at $x = a$, its tangent line there is

$$y = f(a) + f'(a)(x - a).$$

For x near a , the curve and its tangent line are almost on top of each other, so the tangent line's height is a good stand-in for $f(x)$.

Definition 2. The **linear approximation** (or **tangent line approximation**) of f at $x = a$ is the linear function

$$L(x) = f(a) + f'(a)(x - a), \quad \text{and} \quad f(x) \approx L(x) \quad \text{for } x \text{ near } a.$$

L is also called the **linearization** of f at a .

The strategy is always the same: pick a nearby input a where $f(a)$ and $f'(a)$ are easy, build $L(x)$, and evaluate it at the point you actually want.

Example 4 (Estimating $\sqrt{101}$ by hand). Use a linear approximation of $f(x) = \sqrt{x}$ to estimate $\sqrt{101}$.

Solution. The nearest input with an easy square root is $a = 100$, since $\sqrt{100} = 10$. We need $f(a)$ and $f'(a)$:

$$f(x) = \sqrt{x} \Rightarrow f(100) = 10, \quad f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(100) = \frac{1}{2 \cdot 10} = \frac{1}{20}.$$

The linearization at $a = 100$ is

$$L(x) = 10 + \frac{1}{20}(x - 100).$$

Evaluate at $x = 101$:

$$\sqrt{101} \approx L(101) = 10 + \frac{1}{20}(101 - 100) = 10 + \frac{1}{20} = 10.05.$$

A calculator gives 10.0499...; we are off by about one ten-thousandth — and we did it in our heads. Knowing the *rate* $f'(100) = \frac{1}{20}$ told us how much the square root moves when the input nudges up by 1.

Example 5 (Linear approximation of a cube root). Find the linear approximation of $f(x) = \sqrt[3]{x}$ at $x = 8$ and use it to estimate $\sqrt[3]{8.1}$.

Solution. Take $a = 8$, where $\sqrt[3]{8} = 2$. Writing $f(x) = x^{1/3}$,

$$f(8) = 2, \quad f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}} \Rightarrow f'(8) = \frac{1}{3 \cdot 4} = \frac{1}{12}.$$

So

$$L(x) = 2 + \frac{1}{12}(x - 8), \quad \sqrt[3]{8.1} \approx L(8.1) = 2 + \frac{1}{12}(0.1) = 2 + \frac{1}{120} \approx 2.00833.$$

The tangent line at a convenient perfect cube does the work.

A trigonometric version works the same way. To approximate $\sin$ near a familiar angle, we linearize at that angle. For instance, near $a = \frac{\pi}{3}$ (which is 60°), with $f(x) = \sin x$ and $f'(x) = \cos x$,

$$L(x) = \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right),$$

which estimates $\sin$ for angles near 60° . (Remember to work in radians.)

Practice (work this out). Find the linearization of $f(x) = \sqrt{x}$ at $x = 9$ and use it to estimate $\sqrt{9.1}$.

Differentials

Linear approximation also estimates how much the *output* moves when the input moves a little. We give that change a name.

Definition 3. Let $y = f(x)$ be differentiable. Let dx be an independent variable (any nonzero real number, thought of as a small change in x). The **differential** dy is defined by

$$dy = f'(x) dx.$$

If x changes from a to $a + dx$, the *true* change in y is $\Delta y = f(a + dx) - f(a)$. The tangent line predicts a change of $dy = f'(a) dx$, and for small dx these are close:

$$\Delta y \approx dy = f'(a) dx.$$

This is the linear approximation rewritten in terms of changes.

Example 6 (Approximating a change with a differential). Let $y = x^2 + 2x$. Compute the actual change Δy and the differential dy as x goes from 3 to 3.1.

Solution. Here $a = 3$ and $dx = 0.1$. The actual change is

$$\Delta y = f(3.1) - f(3) = [(3.1)^2 + 2(3.1)] - [3^2 + 2(3)] = 15.81 - 15 = 0.81.$$

For the differential, $f'(x) = 2x + 2$, so

$$dy = f'(3) dx = (2(3) + 2)(0.1) = 8(0.1) = 0.8.$$

The estimate $dy = 0.8$ is within 0.01 of the true change 0.81 — and it took only one derivative, no re-evaluating the function.

Estimating Error with Differentials

Measurements are never exact, and a small error in what we *measure* produces an error in whatever we *compute* from it. Differentials estimate that spread. If the exact input is a but the measured value is $a+dx$, the error in f is the **propagated error** $\Delta y = f(a+dx) - f(a)$, which we approximate by $dy = f'(a) dx$.

Example 7 (Error in the volume of a cube). The side of a cube is measured as 5 cm, accurate to within 0.1 cm. Use differentials to estimate the error in the computed volume.

Solution. The volume is $V = x^3$, so $dV = 3x^2 dx$. The measurement is accurate to ± 0.1 cm, meaning $-0.1 \leq dx \leq 0.1$. With the measured side $x = 5$,

$$dV = 3(5)^2 dx = 75 dx, \quad -75(0.1) \leq dV \leq 75(0.1), \quad -7.5 \leq dV \leq 7.5 \text{ cm}^3.$$

So a 0.1-cm uncertainty in the side produces roughly a $\pm 7.5 \text{ cm}^3$ uncertainty in the volume. (Computing the exact volumes at 4.9 and 5.1 gives changes of -7.351 and $+7.651$, so the differential estimate is very close.)

Once we have the absolute error we can report it *relative* to the size of the quantity. The **relative error** is $\frac{\Delta q}{q}$, where q is the actual value, and the **percentage error** is the relative error written as a percent. For the cube above, the relative error in the volume is about $\frac{7.5}{125} = 0.06$, a 6% error — notice how a tiny 2% error in the side ($\frac{0.1}{5}$) roughly triples in the volume, because V depends on the *cube* of the side.

Practice (work this out). The radius of a circle is measured as 10 cm with a possible error of 0.05 cm. Using $A = \pi r^2$ and the differential $dA = 2\pi r dr$, estimate the largest possible error in the computed area, and give the percentage error.

Summary

- **Related rates:** when quantities are linked by an equation, differentiate that equation with respect to t (chain rule) to relate their rates. Assign variables, draw a figure, differentiate, and substitute numbers *last*.
- Never substitute a value for a changing quantity before differentiating — it kills its derivative and breaks the problem.
- Use similar triangles (as in the cone) to cut a related-rates problem down to a single variable before differentiating.
- **Linear approximation:** near $x = a$, $f(x) \approx L(x) = f(a) + f'(a)(x - a)$. This estimates values like $\sqrt{101}$ using a nearby easy point.
- **Differentials:** $dy = f'(x) dx$ approximates the change Δy in the output for a small change dx in the input.
- **Error:** a measurement error dx propagates to $dy = f'(a) dx$; report it as absolute error, relative error $\frac{\Delta q}{q}$, or percentage error.
- Next time: maximum and minimum values.