

Maxima and Minima

Section 4.3

Calculus I — Lecture Notes

October 15, 2026

Where does a function reach its highest and lowest points, and what does the derivative tell us about them?

Many optimization questions are max/min questions: the cheapest can, the largest fenced area, the peak height of a thrown ball. Today we find the largest and smallest values of a function, and see how the derivative narrows down where they can occur.

Absolute vs. Local Extrema

Definition 1. Let f be defined on an interval I and let $c \in I$. Then f has an **absolute maximum** on I at c if $f(c) \geq f(x)$ for all $x \in I$, and an **absolute minimum** at c if $f(c) \leq f(x)$ for all $x \in I$. Either one is called an **absolute extremum**.

“Absolute” has nothing to do with absolute value; an absolute extremum can be positive, negative, or zero. For example, $f(x) = x^2 + 1$ has an absolute minimum value of 1 at $x = 0$, but no absolute maximum.

Definition 2. A function f has a **local maximum** at c if there is an open interval I containing c with $f(c) \geq f(x)$ for all $x \in I$; it has a **local minimum** at c if $f(c) \leq f(x)$ for all such x . Either is a **local extremum**.

A local extremum only has to beat nearby values. An endpoint of the interval is not a local extremum (no open interval around it lies in the domain), but it can be an absolute extremum.

Theorem 1 (Extreme Value Theorem). If f is continuous on a closed, bounded interval $[a, b]$, then f attains an absolute maximum and an absolute minimum somewhere on $[a, b]$.

Both hypotheses are needed. On the open interval $(0, 2)$, $f(x) = x$ has no largest value, and a discontinuity can also remove the guarantee.

Example 1 (Reading off extrema). Find the absolute and local extrema of $f(x) = x^2$ on $[-1, 2]$, and then on the open interval $(-1, 2)$.

Solution. On $[-1, 2]$ the graph is a piece of a parabola with its vertex at the origin. The lowest value is $f(0) = 0$, which is both an absolute and a local minimum. The highest value is $f(2) = 4$ at the right endpoint: an absolute maximum, but not a local one. The other endpoint, $f(-1) = 1$, is neither.

On $(-1, 2)$ the minimum 0 at $x = 0$ survives, but there is no absolute maximum: $f(x)$ gets as close to 4 as we like without reaching it. Dropping the closed interval breaks the Extreme Value Theorem’s guarantee.

Critical Points and Fermat's Theorem

At a smooth interior peak or valley the tangent line is horizontal, so $f' = 0$. At a corner, such as the bottom of a “V”, f' does not exist.

Definition 3. Let c be an interior point of the domain of f . We call c a **critical number** (or **critical point**) of f if $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 2 (Fermat's Theorem). If f has a local extremum at c and f is differentiable at c , then $f'(c) = 0$.

Fermat used this idea to find maxima and minima around 1636, about thirty years before Newton and Leibniz.

So **every local extremum occurs at a critical number**. The converse is false: $f(x) = x^3$ has $f'(0) = 0$, but x^3 is increasing and has no extremum at 0.

Example 2 (Finding critical numbers). Find all critical numbers of $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$.

Solution.

$$f'(x) = x^2 - 5x + 4 = (x - 1)(x - 4).$$

f' is a polynomial, so it is defined everywhere, and the critical numbers are the solutions of $f'(x) = 0$:

$$x = 1 \quad \text{and} \quad x = 4.$$

These are only candidates for local extrema; each still has to be checked.

Example 3 (A critical number where the derivative is undefined). Find all critical numbers of $f(x) = x^{2/3}$.

Solution.

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}.$$

$f'(x) = 0$ has no solution. But $f'(0)$ does not exist while $f(0) = 0$ is defined, so $x = 0$ is a critical number. The graph has a cusp there, and since $x^{2/3} \geq 0$, it is an absolute minimum.

Example 4 (Critical numbers of a quotient). Find all critical numbers of $f(x) = \frac{x}{x^2 + 1}$.

Solution. By the quotient rule,

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

The denominator is never 0, so f' exists everywhere. Setting the numerator to zero gives $1 - x^2 = 0$, so the critical numbers are $x = -1$ and $x = 1$.

Practice (work this out). Find all critical numbers of $f(x) = 4x^3 - 3x$.

The Closed-Interval Method

By the Extreme Value Theorem, absolute extrema exist on $[a, b]$. By Fermat's Theorem, an interior extremum is at a critical number. The only other candidates are the endpoints.

Definition 4 (Locating Absolute Extrema on $[a, b]$). Let f be continuous on $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find the critical numbers of f in the open interval (a, b) and evaluate f at each.
3. Compare all the values from steps 1 and 2. The largest is the absolute maximum; the smallest is the absolute minimum.

Example 5 (Absolute extrema of a quadratic). Find the absolute maximum and minimum of $f(x) = -x^2 + 3x - 2$ on $[1, 3]$, and say where they occur.

Solution.

Step 1 (endpoints).

$$f(1) = -(1) + 3 - 2 = 0, \quad f(3) = -9 + 9 - 2 = -2.$$

Step 2 (critical numbers). $f'(x) = -2x + 3 = 0$ when $x = \frac{3}{2}$, which lies in $(1, 3)$.

$$f\left(\frac{3}{2}\right) = -\frac{9}{4} + \frac{9}{2} - 2 = \frac{1}{4}.$$

Step 3 (compare).

x	$f(x)$	conclusion
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	absolute maximum
3	-2	absolute minimum

The absolute maximum is $\frac{1}{4}$ at $x = \frac{3}{2}$, and the absolute minimum is -2 at $x = 3$.

Example 6 (When an endpoint wins). Find the absolute extrema of $f(x) = x^3 - 3x$ on $[0, 3]$.

Solution. **Step 1.** $f(0) = 0$ and $f(3) = 27 - 9 = 18$.

Step 2. $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$, so $f'(x) = 0$ at $x = 1$ and $x = -1$. Only $x = 1$ lies in $(0, 3)$, and

$$f(1) = 1 - 3 = -2.$$

Step 3. Compare $f(0) = 0$, $f(1) = -2$, $f(3) = 18$. The absolute maximum is 18 at the endpoint $x = 3$, and the absolute minimum is -2 at $x = 1$. The maximum is at an endpoint, which is why the method checks endpoints.

Example 7 (When the derivative is undefined). Find the absolute extrema of $f(x) = 3x^{2/3} - 2x$ on $[-1, 1]$.

Solution. **Step 1.** $f(-1) = 3(-1)^{2/3} + 2 = 3 + 2 = 5$ and $f(1) = 3 - 2 = 1$.

Step 2.

$$f'(x) = 2x^{-1/3} - 2 = \frac{2}{x^{1/3}} - 2.$$

$f'(x) = 0$ when $x^{1/3} = 1$, so $x = 1$, which is an endpoint and was already evaluated in Step 1. $f'(x)$ is undefined at $x = 0$, which lies in $(-1, 1)$, and $f(0) = 0$.

Step 3. Compare $f(-1) = 5$, $f(0) = 0$, $f(1) = 1$. The absolute maximum is 5 at $x = -1$, and the absolute minimum is 0 at $x = 0$, where f' does not exist. Setting $f' = 0$ alone would have missed it.

Example 8 (A rational function). Find the absolute extrema of $f(x) = \frac{x}{x^2 + 1}$ on $[0, 3]$.

Solution. **Step 1.** $f(0) = 0$ and $f(3) = \frac{3}{10}$.

Step 2. From the earlier example, the critical numbers are $x = \pm 1$. Only $x = 1$ lies in $(0, 3)$, and $f(1) = \frac{1}{2}$.

Step 3. Compare 0, $\frac{1}{2}$, and $\frac{3}{10}$. The absolute maximum is $\frac{1}{2}$ at $x = 1$, and the absolute minimum is 0 at $x = 0$.

Example 9 (A trigonometric function). Find the absolute extrema of $f(x) = x + 2 \cos x$ on $[0, 2\pi]$.

Solution. **Step 1.** $f(0) = 0 + 2 = 2$ and $f(2\pi) = 2\pi + 2 \approx 8.28$.

Step 2. $f'(x) = 1 - 2 \sin x = 0$ when $\sin x = \frac{1}{2}$, so $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Both lie in $(0, 2\pi)$:

$$f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3} \approx 2.26, \quad f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3} \approx 0.89.$$

Step 3. The absolute maximum is $2\pi + 2$ at the endpoint $x = 2\pi$, and the absolute minimum is $\frac{5\pi}{6} - \sqrt{3}$ at $x = \frac{5\pi}{6}$.

Example 10 (The peak of a thrown ball). A ball is thrown upward from a height of 5 ft, and its height after t seconds is $h(t) = -16t^2 + 64t + 5$ for $0 \leq t \leq 4$. What is its maximum height?

Solution. **Step 1.** $h(0) = 5$ and $h(4) = -256 + 256 + 5 = 5$.

Step 2. $h'(t) = -32t + 64 = 0$ when $t = 2$, and $h(2) = -64 + 128 + 5 = 69$.

Step 3. The maximum height is 69 ft, reached at $t = 2$ seconds, which is exactly when the velocity $h'(t)$ is zero.

Practice (work this out). Find the absolute maximum and minimum of $f(x) = x^3 - 6x^2 + 9x$ on $[0, 5]$, and say where each occurs.

Summary

- An **absolute** extremum is highest/lowest over the whole interval; a **local** extremum only beats nearby values. Endpoints can be absolute extrema but not local ones.
- **Extreme Value Theorem:** a continuous function on $[a, b]$ attains an absolute max and min.
- A **critical number** is an interior point where $f'(c) = 0$ or $f'(c)$ does not exist. **Fermat's Theorem:** every local extremum is at a critical number, but not every critical number is an extremum.
- **Closed-interval method:** compare f at the endpoints and at the interior critical numbers.
- Next time: Rolle's Theorem and the Mean Value Theorem.