

The Mean Value Theorem

Section 4.4

Calculus I — Lecture Notes

October 20, 2026

Somewhere between two points, the instantaneous rate of change must equal the average rate.

Last time we used the Extreme Value Theorem and Fermat's Theorem to locate maxima and minima. Today those two results let us prove the Mean Value Theorem, which relates the average rate of change of a function to its instantaneous rate of change.

Rolle's Theorem

Theorem 1 (Rolle's Theorem). Let f be continuous on $[a, b]$ and differentiable on (a, b) , with $f(a) = f(b)$. Then there is at least one $c \in (a, b)$ where $f'(c) = 0$.

Proof sketch. By the Extreme Value Theorem, f has an absolute max and min on $[a, b]$. If both are at the endpoints, then f is constant and $f' = 0$ everywhere. Otherwise one is at an interior point c , and Fermat's Theorem gives $f'(c) = 0$.

Differentiability is needed: $f(x) = |x| - 1$ has $f(-1) = f(1) = 0$, but f' is never 0 on $(-1, 1)$ because of the corner at $x = 0$.

Example 1 (Checking Rolle's Theorem). Verify that $f(x) = x^2 - 4x + 3$ satisfies the hypotheses of Rolle's Theorem on $[1, 3]$, and find the c it guarantees.

Solution. f is a polynomial, so it is continuous on $[1, 3]$ and differentiable on $(1, 3)$. At the endpoints,

$$f(1) = 1 - 4 + 3 = 0, \quad f(3) = 9 - 12 + 3 = 0,$$

so $f(1) = f(3)$. Now solve $f'(c) = 0$:

$$f'(x) = 2x - 4 = 0 \implies c = 2,$$

which lies in $(1, 3)$. The parabola has a horizontal tangent at its vertex $(2, -1)$.

Example 2 (More than one c). Find every c guaranteed by Rolle's Theorem for $f(x) = x^3 - x$ on $[-1, 1]$.

Solution. f is a polynomial, and $f(-1) = -1 + 1 = 0 = f(1)$. Then

$$f'(x) = 3x^2 - 1 = 0 \implies x^2 = \frac{1}{3} \implies c = \pm \frac{1}{\sqrt{3}}.$$

Both values lie in $(-1, 1)$, so there are two such points: a local max at $c = -\frac{1}{\sqrt{3}}$ and a local min at $c = \frac{1}{\sqrt{3}}$. Rolle's Theorem promises *at least* one c , not exactly one.

Example 3 (Exactly one root). Show that $x^3 + 3x + 1 = 0$ has exactly one real solution.

Solution. Let $f(x) = x^3 + 3x + 1$.

At least one. $f(-1) = -1 - 3 + 1 = -3 < 0$ and $f(0) = 1 > 0$. Since f is continuous, the Intermediate Value Theorem gives a root in $(-1, 0)$.

At most one. Suppose there were two roots $a < b$. Then $f(a) = f(b) = 0$, so Rolle's Theorem gives some $c \in (a, b)$ with $f'(c) = 0$. But

$$f'(x) = 3x^2 + 3 \geq 3 > 0 \quad \text{for every } x,$$

so f' is never 0. That contradiction means there cannot be two roots.

The Intermediate Value Theorem finds a root; Rolle's Theorem rules out a second one.

Example 4 (The Laffer curve). The Laffer curve, named for economist Arthur Laffer, models tax revenue as a function of the tax rate. The idea is that beyond some point, raising the rate lowers revenue.

Let $R(t)$ be revenue at tax rate t , $0 \leq t \leq 1$. A 0% tax collects nothing, and so does a 100% tax, so $R(0) = R(1) = 0$. If R is differentiable, Rolle's Theorem gives a rate $c \in (0, 1)$ with $R'(c) = 0$.

A toy model. Let $R(t) = 100t(1 - t)$ (billions of dollars). Then

$$R'(t) = 100(1 - 2t) = 0 \implies c = \frac{1}{2},$$

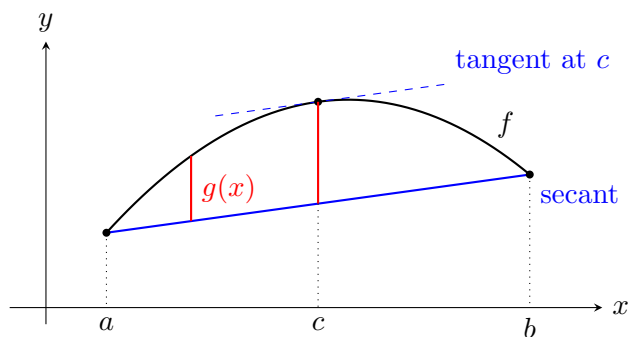
and $R(\frac{1}{2}) = 25$. The theorem guarantees that a peak exists but not where it is; real revenue curves are not symmetric.

The Mean Value Theorem

Theorem 2 (Mean Value Theorem). Let f be continuous on $[a, b]$ and differentiable on (a, b) . Then there is at least one $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $m = \frac{f(b) - f(a)}{b - a}$ and $g(x) = f(x) - [f(a) + m(x - a)]$. The bracket is the secant line in point-slope form, so g is the vertical distance from the graph to the secant line. Then g is continuous on $[a, b]$, differentiable on (a, b) , and $g(a) = g(b) = 0$. By Rolle's Theorem, $g'(c) = 0$ for some $c \in (a, b)$. Since $g'(x) = f'(x) - m$, this gives $f'(c) = m$. $\square$



$g(x)$ is the vertical gap between the graph and the secant. It is 0 at a and b , so by Rolle's Theorem it levels off at some c , where the tangent is parallel to the secant.

The right side is the slope of the **secant line** through $(a, f(a))$ and $(b, f(b))$; the left side is the slope of the **tangent line** at c . So some tangent is parallel to the secant. Rolle's Theorem is the case of a horizontal secant. For example, if you drive 30 miles in half an hour, at some instant your speed was exactly 60 mph.

Example 5 (Verifying the Mean Value Theorem). For $f(x) = \sqrt{x}$ on $[0, 9]$, confirm the hypotheses hold and find every c the theorem guarantees.

Solution. f is continuous on $[0, 9]$ and differentiable on $(0, 9)$. The secant slope is

$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9} = \frac{3}{9} = \frac{1}{3}.$$

Since $f'(x) = \frac{1}{2\sqrt{x}}$,

$$\frac{1}{2\sqrt{c}} = \frac{1}{3} \implies 2\sqrt{c} = 3 \implies \sqrt{c} = \frac{3}{2} \implies c = \frac{9}{4}.$$

Since $\frac{9}{4}$ lies in $(0, 9)$, it is the guaranteed point: the tangent line at $x = \frac{9}{4}$ is parallel to the secant through $(0, 0)$ and $(9, 3)$.

Example 6 (Rejecting a candidate). Find every c guaranteed by the Mean Value Theorem for $f(x) = x^3$ on $[0, 2]$.

Solution. f is a polynomial, so the hypotheses hold. The secant slope is

$$\frac{f(2) - f(0)}{2 - 0} = \frac{8 - 0}{2} = 4.$$

Set $f'(c) = 3c^2$ equal to it:

$$3c^2 = 4 \implies c = \pm \frac{2}{\sqrt{3}} \approx \pm 1.155.$$

Only $c = \frac{2}{\sqrt{3}}$ lies in $(0, 2)$. The negative solution is outside the interval and is thrown out.

Example 7 (The speed camera). A car passes a toll camera at 1:00 PM and a second camera 30 miles down the highway at 1:20 PM. The speed limit is 70 mph. Can the driver be ticketed?

Solution. Let $s(t)$ be the car's position in miles, with t in hours after 1:00. Position is differentiable (its derivative is velocity), and the average velocity over $[0, \frac{1}{3}]$ is

$$\frac{s(\frac{1}{3}) - s(0)}{\frac{1}{3} - 0} = \frac{30}{1/3} = 90 \text{ mph.}$$

By the Mean Value Theorem, at some instant the car's speed was exactly 90 mph. The cameras never saw it speeding, but the theorem proves it was.

Example 8 (Bounding a function). Suppose f is differentiable everywhere, $f(0) = 3$, and $f'(x) \leq 5$ for all x . How large can $f(2)$ be?

Solution. By the Mean Value Theorem on $[0, 2]$, there is a $c \in (0, 2)$ with

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \leq 5.$$

So $f(2) - 3 \leq 10$, and $f(2) \leq 13$. The bound is sharp: $f(x) = 5x + 3$ reaches it.

Example 9 (When a hypothesis fails). Let $f(x) = \frac{1}{x}$ on $[-1, 1]$. Is there a $c \in (-1, 1)$ with $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$?

Solution. The secant slope is $\frac{1 - (-1)}{2} = 1$. But $f'(x) = -\frac{1}{x^2} < 0$ for every $x \neq 0$, so $f'(c) = 1$ never happens. The theorem does not apply because f is not continuous on $[-1, 1]$: it is undefined at $x = 0$.

Practice (work this out). Verify that $f(x) = 2x^2 - 8x + 6$ satisfies the hypotheses of the Mean Value Theorem on $[1, 3]$, then find every $c \in (1, 3)$ it guarantees.

Consequences of the Mean Value Theorem

Recall that f is **increasing** on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, and **decreasing** if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

- **Zero derivative means constant.** If $f'(x) = 0$ on an interval, then f is constant there. (For any two points, the secant slope equals some $f'(c) = 0$.)
- **Equal derivatives differ by a constant.** If $f'(x) = g'(x)$ on an interval, then $f(x) = g(x) + C$. (Apply the previous fact to $f - g$.)
- **Sign of the derivative gives monotonicity.** If $f'(x) > 0$ on an interval, f is increasing there; if $f'(x) < 0$, f is decreasing. (Between any two points, the secant slope equals a tangent slope of the same sign.)

The third fact is the starting point for next lecture.

Example 10 (A trig identity from calculus). Show that $\sin^2 x + \cos^2 x = 1$ for all x using only derivatives.

Solution. Let $f(x) = \sin^2 x + \cos^2 x$. By the chain rule,

$$f'(x) = 2 \sin x \cos x + 2 \cos x (-\sin x) = 0.$$

A zero derivative means f is constant, so $f(x) = f(0) = 0 + 1 = 1$ for every x .

Example 11 (Recovering a function from its derivative). Find every function f with $f'(x) = 2x$, and the one with $f(1) = 5$.

Solution. We know $g(x) = x^2$ has $g'(x) = 2x$. Equal derivatives differ by a constant, so every such f is

$$f(x) = x^2 + C.$$

From $f(1) = 1 + C = 5$ we get $C = 4$, so $f(x) = x^2 + 4$. This idea returns in Chapter 5 as the antiderivative.

Practice (work this out). Show that $x^5 + x - 1 = 0$ has exactly one real solution. (Use the Intermediate Value Theorem on $[0, 1]$, then Rolle's Theorem.)

Summary

- **Rolle's Theorem:** if $f(a) = f(b)$, some interior c has $f'(c) = 0$.
- **Exactly one root:** the Intermediate Value Theorem gives a root; if f' is never 0, Rolle's Theorem rules out a second.
- **Mean Value Theorem:** some interior c has $f'(c) = \frac{f(b) - f(a)}{b - a}$.
- **Consequences:** $f' = 0 \Rightarrow$ constant; $f' = g' \Rightarrow$ differ by a constant; sign of f' gives increasing/decreasing.
- Next time: using f' and f'' to classify critical points and sketch graphs.