

Graphs and Infinity

Sections 4.5 & 4.6

Calculus I — Lecture Notes

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The derivative already knows the shape of the graph — we just have to ask it.

By now we can compute derivatives quickly. Today we cash them in. The sign of the **first** derivative tells us where a graph rises and falls; the sign of the **second** derivative tells us which way it curves. Add one more idea — what happens as x runs off to $\pm\infty$ — and we can sketch an accurate picture of almost any function without plotting a hundred points. That is the whole payoff of the last few weeks.

Increasing, Decreasing, and the First Derivative Test

Recall a **critical point** of f is a point c in the domain where $f'(c) = 0$ or $f'(c)$ does not exist. Local maxima and minima can only happen at critical points — but not every critical point is one (think of $f(x) = x^3$ at $x = 0$). To tell which is which, we watch the *sign* of f' .

Definition 1. A function f is **increasing** on an interval I if its graph rises left to right, and **decreasing** on I if it falls. The link to the derivative: if $f' > 0$ throughout I then f is increasing on I ; if $f' < 0$ throughout I then f is decreasing on I .

So f can only switch between rising and falling at a critical point. This gives a clean test.

First Derivative Test. Let c be a critical point of a continuous function f .

- If f' changes from $+$ to $-$ as x increases through c , then f has a **local maximum** at c .
- If f' changes from $-$ to $+$ as x increases through c , then f has a **local minimum** at c .
- If f' does *not* change sign at c , then c is neither.

The procedure: find all critical points, split the line into intervals at those points, and test the sign of f' at one convenient point inside each interval.

Example 1 (Finding local extrema with the first derivative test). Locate all local extrema of $f(x) = x^3 - 3x^2 - 9x - 1$.

Solution. Differentiate and factor to find the critical points:

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1).$$

So $f'(x) = 0$ at $x = -1$ and $x = 3$. These split the line into three intervals. Pick a test point in each and read off the sign of $f'(x) = 3(x - 3)(x + 1)$:

Interval	Test point	Sign of $3(x - 3)(x + 1)$	f
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	increasing
$(-1, 3)$	$x = 0$	$(+)(-)(+) = -$	decreasing
$(3, \infty)$	$x = 4$	$(+)(+)(+) = +$	increasing

At $x = -1$ the sign flips $+$ $\rightarrow$ $-$, so f has a **local maximum** there, with value $f(-1) = -1 - 3 + 9 - 1 = 4$. At $x = 3$ it flips $-$ $\rightarrow$ $+$, so f has a **local minimum**, with value $f(3) = 27 - 27 - 27 - 1 = -28$.

Example 2 (A critical point that is neither). Locate the local extrema of $f(x) = x^{2/3}(x - 5)$, and note where f' fails to exist.

Solution. Write $f(x) = x^{5/3} - 5x^{2/3}$ and differentiate:

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3} \cdot \frac{x - 2}{x^{1/3}}.$$

Here $f'(x) = 0$ at $x = 2$, and $f'(x)$ *does not exist* at $x = 0$ (the denominator vanishes) — both are critical points. Test the sign of $\frac{x - 2}{x^{1/3}}$ on the three intervals:

Interval	Test point	Sign of $(x - 2)/x^{1/3}$	f
$(-\infty, 0)$	$x = -1$	$(-)/(-) = +$	increasing
$(0, 2)$	$x = 1$	$(-)/(+) = -$	decreasing
$(2, \infty)$	$x = 3$	$(+)/(+) = +$	increasing

At $x = 0$ the sign flips $+ \rightarrow -$: a **local maximum**, even though the graph has a sharp corner there and $f'(0)$ does not exist. At $x = 2$ it flips $- \rightarrow +$: a **local minimum**. Critical points where the derivative is undefined count too.

Concavity and Inflection Points

Rising or falling is only half the story. A graph can rise while *curving up* (like the right side of a parabola) or rise while *curving down*. That bending is **concavity**, and it is governed by the second derivative.

Definition 2. f is **concave up** on an interval where f' is increasing, and **concave down** where f' is decreasing. Since f' increases exactly when its own derivative is positive, the **concavity test** says:

$$f''(x) > 0 \Rightarrow f \text{ concave up}, \quad f''(x) < 0 \Rightarrow f \text{ concave down}.$$

Definition 3. An **inflection point** is a point where f is continuous and its concavity *changes* (up to down, or down to up). These can only occur where $f''(x) = 0$ or $f''(x)$ is undefined — but, as with critical points, such a point is only a candidate: the concavity must actually switch.

Example 3 (Concavity and inflection points). For $f(x) = x^3 - 6x^2 + 9x + 30$, find the intervals of concavity and any inflection points.

Solution. Take two derivatives:

$$f'(x) = 3x^2 - 12x + 9, \quad f''(x) = 6x - 12 = 6(x - 2).$$

Since f'' is defined everywhere, the only candidate is $f''(x) = 0$, i.e. $x = 2$. Test the sign of f'' on each side:

Interval	Test point	Sign of $6(x - 2)$	Concavity
$(-\infty, 2)$	$x = 0$	$-$	concave down
$(2, \infty)$	$x = 3$	$+$	concave up

The concavity changes at $x = 2$, so there is an **inflection point** there. Its height is $f(2) = 8 - 24 + 18 + 30 = 32$, giving the point $(2, 32)$.

The Second Derivative Test

Concavity also gives a shortcut for classifying a critical point. If the graph is concave *up* at a spot where $f' = 0$, it holds water — a minimum; concave *down*, it spills — a maximum.

Second Derivative Test. Suppose $f'(c) = 0$.

- If $f''(c) > 0$, then f has a **local minimum** at c .
- If $f''(c) < 0$, then f has a **local maximum** at c .
- If $f''(c) = 0$, the test is **inconclusive** — fall back on the first derivative test.

Example 4 (Second derivative test). Reclassify the critical points of $f(x) = x^3 - 3x^2 - 9x - 1$ using the second derivative.

Solution. From before, the critical points are $x = -1$ and $x = 3$, and

$$f''(x) = 6x - 6.$$

Evaluate at each: $f''(-1) = -12 < 0$, so $x = -1$ is a **local maximum**; $f''(3) = 12 > 0$, so $x = 3$ is a **local minimum**. This agrees with the sign table from Example 1 — one substitution instead of a whole row of test points.

Limits at Infinity and Horizontal Asymptotes

To finish a graph we need its **end behavior**: what $f(x)$ does as x marches off to $+\infty$ or $-\infty$. If the outputs settle toward a fixed height L , the line $y = L$ is a horizontal guide rail the curve hugs.

Definition 4. If $f(x)$ gets arbitrarily close to a number L as $x \rightarrow \infty$ (or as $x \rightarrow -\infty$), we write $\lim_{x \rightarrow \infty} f(x) = L$ and call the line $y = L$ a **horizontal asymptote** of f . A curve may cross a horizontal asymptote — it only has to *approach* it far out.

The workhorse fact is $\lim_{x \rightarrow \pm\infty} \frac{1}{x^k} = 0$ for any $k > 0$. For a **rational function**, divide numerator and denominator by the highest power of x in the denominator and let those $1/x^k$ terms die. Three outcomes, decided by the degrees:

- degree top = degree bottom: asymptote $y = \frac{\text{leading coeff. top}}{\text{leading coeff. bottom}}$;
- degree top < degree bottom: asymptote $y = 0$;
- degree top > degree bottom: no horizontal asymptote (the graph runs off to $\pm\infty$).

Example 5 (End behavior of rational functions). Find the horizontal asymptotes (if any) of each function.

- (a) $f(x) = \frac{3x - 1}{2x + 5}$
(b) $f(x) = \frac{3x^2 + 2x}{4x^3 - 5x + 7}$
(c) $f(x) = \frac{3x^2 + 4x}{x + 2}$

Solution.

- (a) The denominator's top power is x . Divide through by x :

$$\lim_{x \rightarrow \pm\infty} \frac{3x - 1}{2x + 5} = \lim_{x \rightarrow \pm\infty} \frac{3 - \frac{1}{x}}{2 + \frac{5}{x}} = \frac{3 - 0}{2 + 0} = \frac{3}{2}.$$

Horizontal asymptote $y = \frac{3}{2}$ (equal degrees, ratio of leading coefficients).

(b) Highest power below is x^3 . Divide through by x^3 :

$$\lim_{x \rightarrow \pm\infty} \frac{3x^2 + 2x}{4x^3 - 5x + 7} = \lim_{x \rightarrow \pm\infty} \frac{\frac{3}{x} + \frac{2}{x^2}}{4 - \frac{5}{x^2} + \frac{7}{x^3}} = \frac{0}{4} = 0.$$

Horizontal asymptote $y = 0$ (top degree smaller).

(c) Top degree (2) exceeds bottom degree (1), so there is *no* horizontal asymptote. Long division reveals what the graph does instead:

$$\frac{3x^2 + 4x}{x + 2} = 3x - 2 + \frac{4}{x + 2}.$$

Since $\frac{4}{x+2} \rightarrow 0$, the graph approaches the slanted line $y = 3x - 2$. This is an **oblique asymptote**.

Roots behave the same way, using $\sqrt{x^2} = |x|$ to pull the highest power out. For $f(x) = \frac{3x - 2}{\sqrt{4x^2 + 5}}$, dividing by $|x|$ gives $\lim_{x \rightarrow \infty} f(x) = \frac{3}{\sqrt{4}} = \frac{3}{2}$ while $\lim_{x \rightarrow -\infty} f(x) = -\frac{3}{2}$ — *two* horizontal asymptotes, one at each end.

Putting It Together: Curve Sketching

Now we have every tool. Here is the checklist for graphing an unfamiliar f :

1. **Domain** and **intercepts** (solve $f(x) = 0$ and compute $f(0)$).
2. **End behavior**: limits as $x \rightarrow \pm\infty$; horizontal/oblique asymptotes.
3. **Vertical asymptotes** (where a denominator vanishes).
4. f' : increasing/decreasing intervals and local extrema.
5. f'' : concavity intervals and inflection points.

Example 6 (Sketching a polynomial). Sketch $f(x) = (x - 1)^2(x + 2)$.

Solution. Work the checklist.

Domain and intercepts. A polynomial, so the domain is all of $\mathbb{R}$. Setting $f(x) = 0$ gives x -intercepts $(1, 0)$ (a double root) and $(-2, 0)$; the y -intercept is $f(0) = (1)(2) = 2$.

End behavior. Multiplying out, $f(x) = x^3 - 3x + 2$, so the leading term is x^3 . Thus $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$. No horizontal or vertical asymptotes.

First derivative. $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$, so the critical points are $x = \pm 1$. Sign of $3(x - 1)(x + 1)$:

Interval	Test point	Sign	f
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	increasing
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	decreasing
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	increasing

Local maximum at $x = -1$ with $f(-1) = (-2)^2(1) = 4$; local minimum at $x = 1$ with $f(1) = 0$ (the double root just kisses the axis).

Second derivative. $f''(x) = 6x$, zero at $x = 0$. It is negative for $x < 0$ (concave down) and positive for $x > 0$ (concave up), so $(0, 2)$ is an inflection point.

Plot the intercepts $(-2, 0)$, $(1, 0)$, $(0, 2)$, the local max $(-1, 4)$ and local min $(1, 0)$, draw the curve concave down then concave up through $(0, 2)$, and let the ends run off like x^3 .

Practice (work this out). Use the first derivative test to find all local extrema of $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Practice (work this out). For $f(x) = \frac{2x^2 - 1}{3x^2 + 5}$, find all horizontal asymptotes, then find the intervals of concavity and any inflection points of $g(x) = x^3 - 4x^2 + x + 2$.

Summary

- **First derivative:** $f' > 0$ means increasing, $f' < 0$ means decreasing. Local extrema live at critical points, where $f' = 0$ or is undefined.
- **First Derivative Test:** a sign change of f' at c gives a max ($+ \rightarrow -$) or a min ($- \rightarrow +$); no change means neither.
- **Second derivative:** $f'' > 0$ concave up, $f'' < 0$ concave down. Concavity changes only at **inflection points** (where $f'' = 0$ or undefined *and* the sign actually flips).
- **Second Derivative Test:** at a critical point, $f'' > 0$ min, $f'' < 0$ max, $f'' = 0$ inconclusive.
- **Limits at infinity** give horizontal asymptotes: for a rational function compare the degrees — equal (ratio of leading coefficients), top smaller ($y = 0$), top larger (none, possibly an oblique asymptote).
- **Curve sketching:** domain and intercepts, end behavior and asymptotes, then f' and f'' assemble an accurate graph.