

Curve Sketching and Applied Optimization

Sections 4.5 & 4.7

Calculus I — Lecture Notes

October 27, 2026

Curve Sketching, Continued

We finish the checklist from last time with two full examples: a polynomial whose critical points hide a surprise, and a trigonometric function on a closed interval.

Example 1 (A quartic with a flat inflection point). Sketch $f(x) = x^4 - 4x^3$.

Solution. Work the checklist.

Domain and intercepts. A polynomial, so the domain is all of $\mathbb{R}$. Factoring, $f(x) = x^3(x - 4)$, so the x -intercepts are $(0, 0)$ and $(4, 0)$, and the y -intercept is $(0, 0)$.

End behavior. The leading term is x^4 , so $f(x) \rightarrow \infty$ as $x \rightarrow \pm\infty$. No asymptotes.

First derivative.

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

so the critical points are $x = 0$ and $x = 3$. Sign of $4x^2(x - 3)$:

Interval	Test point	Sign	f
$(-\infty, 0)$	$x = -1$	$(+)(+)(-) = -$	decreasing
$(0, 3)$	$x = 1$	$(+)(+)(-) = -$	decreasing
$(3, \infty)$	$x = 4$	$(+)(+)(+) = +$	increasing

At $x = 0$ the sign does *not* change ($- \rightarrow -$): the tangent is horizontal, but there is no max or min. The factor x^2 is a square, so it never changes sign. At $x = 3$ the sign goes $- \rightarrow +$: a **local minimum**, $f(3) = 81 - 108 = -27$. Since $f \rightarrow \infty$ at both ends, this is also the absolute minimum.

Second derivative.

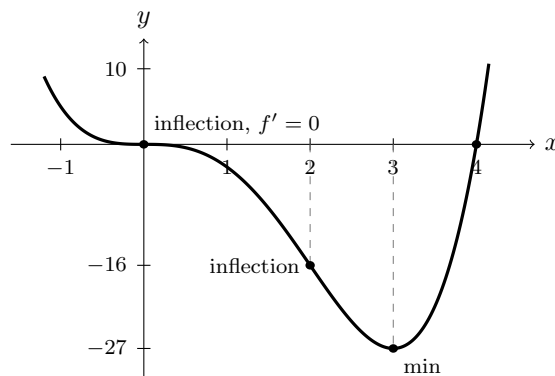
$$f''(x) = 12x^2 - 24x = 12x(x - 2),$$

zero at $x = 0$ and $x = 2$. Sign of $12x(x - 2)$:

Interval	Test point	Sign	f
$(-\infty, 0)$	$x = -1$	$(+)(-)(-) = +$	concave up
$(0, 2)$	$x = 1$	$(+)(+)(-) = -$	concave down
$(2, \infty)$	$x = 3$	$(+)(+)(+) = +$	concave up

The concavity flips at both, so the inflection points are $(0, 0)$ and $(2, f(2)) = (2, 16 - 32) = (2, -16)$.

Assemble. The point $(0, 0)$ is both a critical point and an inflection point: the curve comes down, flattens to a horizontal tangent, and keeps going down, switching from concave up to concave down as it passes through. It bends back up at $(2, -16)$, bottoms out at $(3, -27)$, and climbs through $(4, 0)$.



Note what the Second Derivative Test does here. At $x = 3$, $f''(3) = 36 > 0$ confirms the minimum. At $x = 0$, $f''(0) = 0$, and the test says nothing; only the First Derivative Test tells us $x = 0$ is neither a max nor a min.

Practice (work this out). Sketch $f(x) = x^4 + 4x^3$. (Its derivative factors as $4x^2(x + 3)$ and its second derivative as $12x(x + 2)$.)

Before the second example, a warm-up we already know the answer to. For $f(x) = \sin x$, we get $f'(x) = \cos x$ and $f''(x) = -\sin x = -f(x)$. So sine is concave down exactly where it is above the axis and concave up where it is below, and its inflection points are its zeros. The calculus recovers the familiar wave. Now add a line to it.

Example 2 (A sine wave riding on a line). Sketch $f(x) = x + 2\sin x$ on $[0, 2\pi]$, and find its absolute extrema.

Solution.

Domain and endpoints. The domain is the closed interval $[0, 2\pi]$, with $f(0) = 0$ and $f(2\pi) = 2\pi$.

First derivative.

$$f'(x) = 1 + 2\cos x = 0 \implies \cos x = -\frac{1}{2} \implies x = \frac{2\pi}{3}, \frac{4\pi}{3}.$$

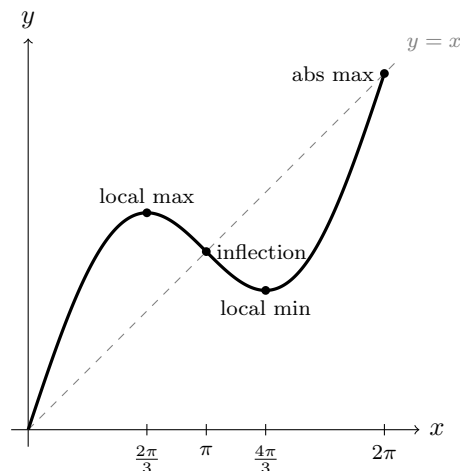
Signs: at $x = 0$, $f' = 3 > 0$; at $x = \pi$, $f' = -1 < 0$; at $x = 2\pi$, $f' = 3 > 0$. So f increases on $[0, \frac{2\pi}{3})$, decreases on $(\frac{2\pi}{3}, \frac{4\pi}{3})$, and increases on $(\frac{4\pi}{3}, 2\pi]$. Using $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$:

$$\text{local max } f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83, \quad \text{local min } f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46.$$

Second derivative. $f''(x) = -2\sin x$. On $(0, \pi)$, $\sin x > 0$, so $f'' < 0$ and f is concave down; on $(\pi, 2\pi)$, $f'' > 0$ and f is concave up. The inflection point is $(\pi, f(\pi)) = (\pi, \pi)$.

Absolute extrema. This is a closed interval, so compare the critical values with the endpoints: 0, 3.83, 2.46, and $2\pi \approx 6.28$. The absolute maximum is 2π at the right endpoint and the absolute minimum is 0 at the left endpoint. The local max and min are real bumps, but neither is the highest or lowest point overall.

Assemble. Since $-2 \leq 2\sin x \leq 2$, the graph stays within 2 of the line $y = x$, crossing it wherever $\sin x = 0$: at 0, π , and 2π .



Applied Optimization

The derivative doesn't just describe change — it finds the best possible answer.

We have spent weeks learning to find where a function is largest or smallest. Today we cash that skill in. Real questions come phrased in words, not formulas: enclose the most garden with the fencing you own, build a box that holds the most, use the least cardboard for a required volume, get as close as possible to a target. In every case the recipe is the same — turn the words into a function of *one* variable, then hunt for its extreme value with the tools of Chapter 4. The only new skill is the translation.

The Strategy

The hard part of an applied problem is almost never the calculus — it is setting it up. Here is the routine we will follow every single time.

Definition 1 (Solving Optimization Problems).

1. Introduce all variables. If it helps, draw a figure and label everything.
2. Decide which quantity is to be maximized or minimized.
3. Write a formula for that quantity. It may involve several variables.
4. Use the relationships between the variables (the **constraint**) to rewrite the quantity as a function of a *single* variable.
5. Identify the domain of that variable that makes physical sense.
6. Locate the maximum or minimum, usually by finding critical points and checking endpoints.

Two ideas from earlier do the heavy lifting in Step 6. When the sensible domain is a **closed, bounded interval** $[a, b]$, the Extreme Value Theorem guarantees a maximum and a minimum, and each occurs either at a critical point or at an endpoint — so we just compare those candidates. When the domain is open or unbounded, we instead reason about the function's behavior at the ends to argue that a lone interior critical point must be the extreme value.

Maximizing Area with Fixed Fencing

The classic first problem: you have a fixed amount of fence and want to enclose as much area as possible. Making one side longer forces the other shorter, so there is a genuine best choice.

Example 3 (A garden against a wall). A rectangular garden is built using a rock wall as one side and 100 ft of wire fencing for the other three sides. What dimensions give the maximum area, and what is that area?

Solution. Let x be the length of each side perpendicular to the wall, and y the length of the side parallel to the wall (the side opposite the wall). The quantity to maximize is the area

$$A = x \cdot y.$$

Only three sides are fenced — two of length x and one of length y — so the constraint is

$$2x + y = 100 \implies y = 100 - 2x.$$

Substitute to get area as a function of one variable:

$$A(x) = x(100 - 2x) = 100x - 2x^2.$$

Both sides must be positive, so $x > 0$ and $y = 100 - 2x > 0$, giving $0 < x < 50$. We maximize A over the closed interval $[0, 50]$; since $A = 0$ at both endpoints and $A > 0$ inside, the maximum is at an interior critical point. Differentiate:

$$A'(x) = 100 - 4x = 0 \implies x = 25.$$

Then $y = 100 - 2(25) = 50$, and

$$A(25) = 25 \cdot 50 = 1250 \text{ ft}^2.$$

So the garden is 25 ft deep and 50 ft wide, for a maximum area of 1250 square feet.

Notice the shape of the answer: the fenced side parallel to the wall came out *twice* the depth. That kind of clean relationship is common in these problems and worth noticing as a sanity check.

Practice (work this out). You have 400 ft of fencing to build a rectangular pen (all four sides fenced). What dimensions maximize the enclosed area, and what is the maximum area?

Practice (work this out). Now flip it. A rectangular garden of area 50 m^2 is built against a wall, so only three sides are fenced. What dimensions use the least fencing? (The domain is open, so argue as in the least-material box below.)

Maximizing the Volume of a Box

Next we build something in three dimensions. The trick is the same — one constraint knocks the volume formula down to a single variable.

Example 4 (An open-top box from a square sheet). An open-top box is made from a 12 in. by 12 in. square of cardboard by cutting a small square of side x from each corner and folding up the flaps. What cut size x gives the largest volume?

Solution. After removing a square of side x from each corner and folding up, the height of the box is x , and the base is a square of side $12 - 2x$. The volume is

$$V(x) = x(12 - 2x)^2.$$

We need $x > 0$ and $12 - 2x > 0$, so the sensible domain is $0 < x < 6$; we optimize over $[0, 6]$, where $V = 0$ at both endpoints. Differentiate with the product rule:

$$V'(x) = (12 - 2x)^2 + x \cdot 2(12 - 2x)(-2) = (12 - 2x)[(12 - 2x) - 4x] = (12 - 2x)(12 - 6x).$$

Setting $V'(x) = 0$ gives $x = 6$ (rejected, it is an endpoint with zero volume) or

$$12 - 6x = 0 \implies x = 2.$$

The only interior critical point is $x = 2$, so the maximum is there. The base is $12 - 2(2) = 8$ in. on a side, and

$$V(2) = 2 \cdot 8^2 = 128 \text{ in}^3.$$

Cut a 2-inch square from each corner; the resulting $8 \times 8 \times 2$ box holds a maximum of 128 cubic inches.

Minimizing Material for a Box

Now flip the question. Fix the volume the container must hold, and use as little material as possible — this is exactly what a manufacturer minimizing packaging cost cares about.

Example 5 (Least material for an open-top box). A box with a square base and an *open* top must hold 32 in^3 . What dimensions use the least material (smallest surface area), and what is that minimum area?

Solution. Let x be the side length of the square base and y the height. With no top, the material is the base plus the four sides:

$$S = x^2 + 4xy.$$

The volume fixes the constraint:

$$x^2y = 32 \implies y = \frac{32}{x^2}.$$

Substitute to make S a function of x alone:

$$S(x) = x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x}.$$

Since $x^2y = 32$ we cannot have $x = 0$, so the domain is the open interval $(0, \infty)$. As $x \rightarrow 0^+$ the term $128/x \rightarrow \infty$, and as $x \rightarrow \infty$ the term $x^2 \rightarrow \infty$; a continuous function that blows up at both ends of $(0, \infty)$ must attain a minimum at an interior critical point. Differentiate:

$$S'(x) = 2x - \frac{128}{x^2}.$$

Set $S'(x) = 0$:

$$2x = \frac{128}{x^2} \implies 2x^3 = 128 \implies x^3 = 64 \implies x = 4.$$

Then $y = 32/4^2 = 2$, and

$$S(4) = 4^2 + \frac{128}{4} = 16 + 32 = 48 \text{ in}^2.$$

A base of 4×4 in. and height 2 in. uses the least material: 48 square inches.

The same setup handles a tin can (a cylinder) or a closed box — only the surface-area formula changes. The engine is always: write the surface area, use the fixed volume to eliminate one variable, minimize.

Practice (work this out). A closed rectangular box with a square base must hold 500 in^3 . Writing x for the base side and y for the height, express the total surface area $S = 2x^2 + 4xy$ as a function of x alone. (You need not minimize it.)

Minimizing a Distance

A different flavor: find the point on a curve closest to a given point. The clean move is to minimize the *square* of the distance — it has the same minimizer as the distance itself, and it avoids dragging a square root through the derivative.

Example 6 (Closest point on a parabola). Find the point on the parabola $y = x^2$ that is closest to the point $(3, 0)$.

Solution. A point on the parabola is (x, x^2) . Its distance to $(3, 0)$ is

$$D = \sqrt{(x - 3)^2 + (x^2 - 0)^2}.$$

Minimizing D is the same as minimizing

$$f(x) = (x - 3)^2 + x^4,$$

its square, which is far easier to differentiate. The domain is all of $\mathbb{R}$. Differentiate:

$$f'(x) = 2(x - 3) + 4x^3 = 4x^3 + 2x - 6.$$

Set $f'(x) = 0$, i.e. $4x^3 + 2x - 6 = 0$, or dividing by 2,

$$2x^3 + x - 3 = 0.$$

Testing $x = 1$: $2(1) + 1 - 3 = 0$, so $x = 1$ is a root, and factoring gives

$$2x^3 + x - 3 = (x - 1)(2x^2 + 2x + 3).$$

The quadratic factor has discriminant $2^2 - 4(2)(3) = 4 - 24 = -20 < 0$, so it has no real roots. Thus $x = 1$ is the only critical point. Since $f(x) \rightarrow \infty$ as $x \rightarrow \pm\infty$, this lone critical point is the minimum. The closest point is

$$(1, 1^2) = (1, 1),$$

at distance $D = \sqrt{(1 - 3)^2 + 1^2} = \sqrt{5}$.

Summary

- **Curve sketching:** a critical point where f' does not change sign is neither a max nor a min, and can also be an inflection point ($x^4 - 4x^3$ at 0). On a closed interval, check the endpoints too ($x + 2\sin x$).
- Every applied optimization problem follows one routine: name the variables, write the quantity to optimize, use a **constraint** to reduce it to one variable, find the domain, then locate the extreme value.
- On a **closed, bounded interval**, compare critical points against the endpoints (Extreme Value Theorem). On an open or unbounded domain, use the function's behavior at the ends to justify that an interior critical point is the answer.
- **Maximize area / volume:** the constraint (perimeter, sheet size) turns a two-variable formula into one variable to differentiate.
- **Minimize material / cost:** fix the required volume, write the surface area, and eliminate a variable — same machine, run in reverse.
- **Minimize distance:** minimize the *square* of the distance; it shares the minimizer and keeps the algebra clean.
- The calculus is routine — the skill is the translation from words to a single-variable function.