

Approximating Areas

Section 5.1

Calculus I — Lecture Notes

October 29, 2026

The Area Problem

We know the area of a rectangle. Everything else is a pile of rectangles.

We can find the area of a rectangle, a triangle, or a circle because we have a formula. But what is the area of the region trapped under a *curve* — say, under $y = x^2$ from $x = 0$ to $x = 2$, above the x -axis? There is no formula for that shape. This is the second great problem of calculus (the first was the tangent line), and it has the same cure: **approximate, then take a limit**.

The strategy is old. Archimedes filled a curved region with shapes he *could* measure and added them up, getting closer and closer to the true area. We will do exactly that with rectangles: slice the region into thin vertical strips, cap each strip with a rectangle, add the areas, and then let the strips get infinitely thin. Before we can add up many rectangles cleanly, we need a compact way to write long sums.

Sigma (Summation) Notation

Adding $1 + 2 + 3 + \cdots + 100$ by writing every term is painful. The Greek capital letter Σ (sigma) is a machine for long sums.

Definition 1. The **sigma notation** for a sum is

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \cdots + a_n.$$

Here a_i is the rule for a typical term, and i is the **index**: it starts at the number below Σ and runs up to the number on top. The index is a *dummy variable* — the sum does not depend on what letter we use, so $\sum_{i=1}^n a_i$ and $\sum_{k=1}^n a_k$ mean the same thing.

For example, $\sum_{i=2}^5 i^2 = 2^2 + 3^2 + 4^2 + 5^2 = 54$.

Sigma notation pulls apart in the natural way. For a constant c and sequences a_i, b_i :

$$\sum_{i=1}^n c = nc, \quad \sum_{i=1}^n c a_i = c \sum_{i=1}^n a_i, \quad \sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i.$$

And three closed-form formulas do a huge amount of work for us:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$$

Example 1 (Reading and evaluating a sum). Evaluate $\sum_{i=1}^5 (2i + 1)$.

Solution. Split the sum with the properties above, then use the formulas.

$$\begin{aligned}\sum_{i=1}^5 (2i + 1) &= 2 \sum_{i=1}^5 i + \sum_{i=1}^5 1 = 2 \cdot \frac{5(6)}{2} + 5(1) \\ &= 2(15) + 5 = 35.\end{aligned}$$

Check by hand: $3 + 5 + 7 + 9 + 11 = 35$. ✓

Example 2 (Using the power formulas). Evaluate $\sum_{i=1}^n (i^2 - 6i + 9)$ for $n = 200$.

Solution. This is the expansion of $\sum (i - 3)^2$; break it into three sums.

$$\begin{aligned}\sum_{i=1}^{200} (i^2 - 6i + 9) &= \sum_{i=1}^{200} i^2 - 6 \sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \\ &= \frac{200(201)(401)}{6} - 6 \cdot \frac{200(201)}{2} + 9(200) \\ &= 2,686,700 - 120,600 + 1,800 = 2,567,900.\end{aligned}$$

Practice (work this out). Evaluate $\sum_{i=1}^{10} i^3$ using the closed-form formula, and check that $\sum_{k=1}^{20} (2k + 1) = 440$.

Approximating Area with Rectangles

Now the main event. Let $f(x)$ be continuous and nonnegative on a closed interval $[a, b]$. We want the area A under $y = f(x)$, above the x -axis, between $x = a$ and $x = b$.

Divide $[a, b]$ into n subintervals of equal width. Each has width

$$\Delta x = \frac{b - a}{n},$$

and the division points are $x_0 = a, x_1, x_2, \dots, x_n = b$, where

$$x_i = a + i \Delta x.$$

This equal spacing is called a **regular partition**. On each subinterval we build a rectangle of width Δx ; the only choice left is how tall to make it, and that choice is set by where we sample the height.

Definition 2. The **left-endpoint approximation** L_n uses the height at the *left* edge of each subinterval; the **right-endpoint approximation** R_n uses the height at the *right* edge:

$$L_n = \sum_{i=1}^n f(x_{i-1}) \Delta x, \quad R_n = \sum_{i=1}^n f(x_i) \Delta x.$$

Each product $f(x_{i-1}) \Delta x$ or $f(x_i) \Delta x$ is just height $\times$ width, the area of one rectangle. We add them up.

Example 3 (Left and right sums for $f(x) = x^2$). Approximate the area under $f(x) = x^2$ on $[0, 2]$ using $n = 4$. Compute both L_4 and R_4 .

Solution. First $\Delta x = \frac{2-0}{4} = 0.5$, so the division points are

$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1, \quad x_3 = 1.5, \quad x_4 = 2.$$

The heights $f(x) = x^2$ at these points are 0, 0.25, 1, 2.25, 4.

For L_4 use the left edges x_0, x_1, x_2, x_3 :

$$\begin{aligned} L_4 &= [f(0) + f(0.5) + f(1) + f(1.5)] \Delta x \\ &= (0 + 0.25 + 1 + 2.25)(0.5) = (3.5)(0.5) = 1.75. \end{aligned}$$

For R_4 use the right edges x_1, x_2, x_3, x_4 :

$$\begin{aligned} R_4 &= [f(0.5) + f(1) + f(1.5) + f(2)] \Delta x \\ &= (0.25 + 1 + 2.25 + 4)(0.5) = (7.5)(0.5) = 3.75. \end{aligned}$$

Because f is increasing on $[0, 2]$, the left sum undershoots and the right sum overshoots. The true area is trapped: $1.75 < A < 3.75$.

The two sums straddle the answer, and the gap between them is wide when n is small. Increase n and the rectangles get thinner, hug the curve better, and the estimates tighten. For $f(x) = x^2$ on $[0, 2]$, the values march toward a common number:

n	L_n	R_n
4	1.75	3.75
8	2.19	3.19
32	2.54	2.79

Both columns are closing in on the exact area, which we will pin down to be $\frac{8}{3} \approx 2.667$.

A sum that is guaranteed to sit below the true area is called a **lower sum**: on each strip it uses the *smallest* value of f . An **upper sum** uses the *largest* value and sits above the area. For a monotone function those smallest and largest values are always at the endpoints of the strip.

Example 4 (A lower sum for a decreasing function). Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$ using $n = 4$.

Solution. Here $\Delta x = \frac{2-1}{4} = 0.25$, with subintervals $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, $[1.75, 2]$. This function is *decreasing* on $[1, 2]$, so the shortest rectangle on each strip sits at the *right* endpoint. A lower sum therefore uses right endpoints:

$$\begin{aligned} R_4 &= [f(1.25) + f(1.5) + f(1.75) + f(2)](0.25) \\ &= [(10 - 1.5625) + (10 - 2.25) + (10 - 3.0625) + (10 - 4)](0.25) \\ &= (8.4375 + 7.75 + 6.9375 + 6)(0.25) \\ &= (29.125)(0.25) \approx 7.28. \end{aligned}$$

So $A \geq 7.28$: this is an underestimate. (Watch the logic — for an *increasing* function the right endpoint gives the *tallest* rectangle, but here the function decreases, so right endpoints give the shortest.)

Riemann Sums

There is no reason to sample only at endpoints. We could take the height at the *midpoint* of each strip, or at any point inside it. Let x_i^* be any point chosen in the i -th subinterval $[x_{i-1}, x_i]$.

Definition 3. A **Riemann sum** for f on $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x,$$

where x_i^* is any sample point in the i -th subinterval. Left sums ($x_i^* = x_{i-1}$), right sums ($x_i^* = x_i$), and midpoint sums ($x_i^* = \text{midpoint}$) are all special cases.

The midpoint choice usually gives the best estimate for a given n , because over/under errors on opposite sides of the strip tend to cancel.

Example 5 (Midpoint sum). Approximate the area under $f(x) = x^2$ on $[0, 2]$ using a midpoint Riemann sum with $n = 4$.

Solution. Again $\Delta x = 0.5$, with subintervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$. The midpoints are

$$x_1^* = 0.25, \quad x_2^* = 0.75, \quad x_3^* = 1.25, \quad x_4^* = 1.75.$$

Their squares are 0.0625, 0.5625, 1.5625, 3.0625. Then

$$\sum_{i=1}^4 f(x_i^*) \Delta x = (0.0625 + 0.5625 + 1.5625 + 3.0625)(0.5) = (5.25)(0.5) = 2.625.$$

Compare with $L_4 = 1.75$ and $R_4 = 3.75$: the midpoint value 2.625 is far closer to the true area $\frac{8}{3} \approx 2.667$, even at the same n .

The Limit of Riemann Sums = the Area

Every table we have built points the same way: as n grows, the rectangles thin out and the Riemann sums converge, no matter how we pick the sample points. That limit is what we *define* the area to be.

Definition 4. Let f be continuous and nonnegative on $[a, b]$. The **area under the curve** $y = f(x)$ on $[a, b]$ is

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

For a continuous f this limit exists and gives the same value for every choice of sample points x_i^* .

Let us actually run the limit once, using right endpoints where the algebra is clean.

Example 6 (Exact area as a limit). Find the exact area under $f(x) = x^2$ on $[0, 2]$ by taking the limit of R_n .

Solution. With n subintervals, $\Delta x = \frac{2}{n}$ and the right endpoints are

$$x_i = 0 + i \Delta x = \frac{2i}{n}, \quad \text{so} \quad f(x_i) = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}.$$

Build the right-endpoint sum and pull constants out front:

$$R_n = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \frac{4i^2}{n^2} \cdot \frac{2}{n} = \frac{8}{n^3} \sum_{i=1}^n i^2.$$

Now apply the square-sum formula $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$:

$$R_n = \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{8(n+1)(2n+1)}{6n^2} = \frac{4(n+1)(2n+1)}{3n^2}.$$

Finally let $n \rightarrow \infty$. Expanding, $\frac{(n+1)(2n+1)}{n^2} = \frac{2n^2 + 3n + 1}{n^2} = 2 + \frac{3}{n} + \frac{1}{n^2} \rightarrow 2$, so

$$A = \lim_{n \rightarrow \infty} R_n = \frac{4}{3} \cdot 2 = \frac{8}{3}.$$

That is the exact area — $2.6\bar{6}$ — and it is precisely where both columns of our earlier table were heading.

Practice (work this out). Let $f(x) = 4 - x^2$ on $[0, 2]$. (a) Compute L_4 and R_4 . (b) Set up the right-endpoint sum R_n with $x_i = \frac{2i}{n}$, simplify it using the sum formulas, and take $n \rightarrow \infty$ to find the exact area.

Summary

- The **area problem**: find the area under a curve, where no geometry formula exists. Fill the region with rectangles, add, then take a limit.
- **Sigma notation** $\sum_{i=1}^n a_i$ compactly writes long sums; it splits over $+$, $-$, and constants, and the closed forms for $\sum i$, $\sum i^2$, $\sum i^3$ collapse them.
- A **regular partition** of $[a, b]$ has width $\Delta x = \frac{b-a}{n}$ and points $x_i = a + i \Delta x$.
- **Left** (L_n) and **right** (R_n) sums cap each strip at its left or right edge; for a monotone f they bracket the true area.
- A **Riemann sum** $\sum f(x_i^*) \Delta x$ samples at any point x_i^* in each strip; the **midpoint** choice is usually most accurate for a given n .
- The **area** is the limit $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$; for $f(x) = x^2$ on $[0, 2]$ it equals $\frac{8}{3}$.
- Next time: this limit becomes the **definite integral**.