

Math 1231 — Quiz 2 (Version A)

Derivatives Sections 3.1–3.5

September 25, 2026 12 points

SOLUTIONS

Problem 1. The derivative as a limit.

(6 points)

(a) (2 points) Let $f(x) = |x - 2|$. Circle the statement that is true at $x = 2$.

- (A) f is differentiable at 2, and $f'(2) = 1$. (B) f is continuous at 2 but not differentiable at 2. ←
(C) f is differentiable at 2, and $f'(2) = 0$. (D) f is neither continuous nor differentiable at 2.

Solution. The graph of $y = |x - 2|$ is a V with its corner at $(2, 0)$. There is no break there, so f is continuous: $\lim_{x \rightarrow 2} f(x) = 0 = f(2)$.

Differentiability fails because the two one-sided difference quotients disagree. For $h > 0$, $|h| = h$; for $h < 0$, $|h| = -h$:

$$\lim_{h \rightarrow 0^+} \frac{|2 + h - 2| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1, \quad \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1.$$

The one-sided limits are 1 and -1 , so the two-sided limit defining $f'(2)$ does not exist. Answer: **(B)**.

This is the standard warning: differentiability implies continuity, but *continuity does not imply differentiability*. A corner is continuous and still has no tangent line.

(b) (2 points) Suppose f is differentiable on all of $\mathbb{R}$. Circle the correct value of

$$\lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h}.$$

- (A) $f(2)$ (B) $f'(x)$
(C) 0 (D) $f'(2)$ ←
(E) the slope of the secant line through $(2, f(2))$ and $(2 + h, f(2 + h))$ (F) the limit does not exist

Solution. No function is given and none is needed. This expression *is* the definition of the derivative at a point:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}, \quad \text{here with } a = 2.$$

Since f is differentiable everywhere, that limit exists and equals $f'(2)$. Answer: **(D)**.

Why each wrong option is tempting:

- **(A)** $f(2)$ — the *value* of the function at 2, not the slope of its tangent there.
- **(B)** $f'(x)$ — the derivative *function*. The limit here is a single number: the input was pinned to 2 before the limit was taken, and the variable being sent to 0 is h , not x .
- **(C)** 0 — the numerator does approach 0, but so does the denominator; $\frac{0}{0}$ is indeterminate, not 0.
- **(E)** — that is the quotient *before* the limit is taken. The secant slopes are what we take the limit of; the limit itself is the tangent slope.
- **(F)** — $\frac{0}{0}$ signals that a limit is waiting to be computed, not that it fails to exist. The hypothesis that f is differentiable at 2 guarantees it exists.

(c) (2 points) Let $f(x) = x^2$. Find $f'(x)$ using the definition of the derivative as a limit. *No derivative rules.* Show each step.

Solution. Step 1. Build $f(x + h)$ by substituting $x + h$ everywhere x appears, then expand:

$$f(x + h) = (x + h)^2 = x^2 + 2xh + h^2.$$

Step 2. Form the difference quotient. The x^2 terms cancel — that is what makes the rest possible:

$$\frac{f(x + h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2) - x^2}{h} = \frac{2xh + h^2}{h}.$$

Step 3. Factor h out of the numerator and cancel it. Substituting $h = 0$ before this step would give $\frac{0}{0}$; after it, the expression is harmless:

$$\frac{2xh + h^2}{h} = \frac{h(2x + h)}{h} = 2x + h.$$

Step 4. Take the limit:

$$f'(x) = \lim_{h \rightarrow 0} (2x + h) = 2x.$$

This agrees with the power rule, $\frac{d}{dx}x^2 = 2x$ — but the rule is what we are *deriving* here, so the limit work is the answer.

Problem 2. Differentiation rules.

(6 points)

(a) (2 points) A particle's position at time t seconds is $s(t) = t^3 - 6t^2 + 9t$ meters. Circle its *velocity* at $t = -1$.

- (A) -16 m/s (B) 24 m/s $\leftarrow$ (C) 18 m/s (D) 0 m/s

Solution. Velocity is the derivative of position, $v(t) = s'(t)$. Differentiate term by term with the power rule:

$$s(t) = t^3 - 6t^2 + 9t \implies v(t) = s'(t) = 3t^2 - 12t + 9.$$

Now evaluate at $t = -1$, watching the signs: $(-1)^2 = 1$ and $-12(-1) = +12$.

$$v(-1) = 3(1) - 12(-1) + 9 = 3 + 12 + 9 = 24 \text{ m/s}.$$

Answer: **(B)**. (Choice (A) is $s(-1) = -1 - 6 - 9 = -16$, the particle's *position* rather than its velocity — the most common mistake here. Choice (C) comes from differentiating $-6t^2$ as $-6t$ instead of $-12t$, giving $3 + 6 + 9 = 18$.)

(b) (2 points) Let $f(x) = \frac{x^2}{x+1}$. Circle an equation of the line tangent to the graph of f at $x = 1$.

- (A) $y - \frac{1}{2} = -\frac{3}{4}(x - 1)$ (B) $y - 1 = \frac{3}{4}\left(x - \frac{1}{2}\right)$
 (C) $y - \frac{1}{2} = \frac{3}{4}(x - 1) \leftarrow$ (D) $y - \frac{1}{2} = \frac{1}{2}(x - 1)$

Solution. A tangent line needs two things: a *point* and a *slope*.

The point. Evaluate f at $x = 1$:

$$f(1) = \frac{1^2}{1+1} = \frac{1}{2}, \quad \text{so the point is } \left(1, \frac{1}{2}\right).$$

The slope. This is a quotient. The quotient rule says: derivative of the top times the bottom, minus the top times the derivative of the bottom, all over the bottom squared. Here the top is x^2 and the bottom is $x + 1$:

$$f'(x) = \frac{(2x)(x+1) - (x^2)(1)}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}, \quad f'(1) = \frac{1+2}{2^2} = \frac{3}{4}.$$

Assemble. Substitute the point and the slope into point-slope form $y - y_0 = m(x - x_0)$:

$$y - \frac{1}{2} = \frac{3}{4}(x - 1).$$

Answer: **(C)**. (Simplifying to $y = \frac{3}{4}x - \frac{1}{4}$ is fine but not required.)

Why each wrong option is tempting:

- **(A)** the quotient-rule numerator written backwards — top times the derivative of the bottom *first* — giving slope $-\frac{3}{4}$. Order matters in the quotient rule; it does not in the product rule, which is why this slip is so common.
- **(B)** the coordinates of the point swapped — x_0 and y_0 traded places. The x -value 1 belongs with x , and $f(1) = \frac{1}{2}$ with y .
- **(D)** used the function value $f(1) = \frac{1}{2}$ as the *slope*. This line does pass through $(1, \frac{1}{2})$, so checking the point alone will not catch the error — the slope has to come from f' .

(c) (2 points) Let $h(x) = \sin(2x)$. The chain rule is not allowed. You may use the identity $\sin(2x) = 2 \sin x \cos x$.

(i) Rewrite h using the identity, then find $h'(x)$ by the product rule.

(ii) Evaluate $h'\left(\frac{\pi}{4}\right)$.

Solution. Step 1. The double-angle identity turns the composition into a product, which we *can* differentiate:

$$h(x) = \sin(2x) = 2 \sin x \cos x.$$

Step 2. Apply the product rule to $\sin x \cos x$, keeping the constant 2 out front:

$$h'(x) = 2 \left[(\cos x)(\cos x) + (-\sin x)(\sin x) \right] = 2(\cos^2 x - \sin^2 x).$$

Step 3. Evaluate at $x = \frac{\pi}{4}$, where $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$:

$$h'\left(\frac{\pi}{4}\right) = 2 \left(\left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 \right) = 2 \left(\frac{1}{2} - \frac{1}{2} \right) = 0.$$

The answer 0 is correct and worth a sanity check: $\sin(2x)$ reaches its maximum value of 1 at $x = \frac{\pi}{4}$, and the tangent line at a peak is horizontal. *Remark.* Since $\cos^2 x - \sin^2 x = \cos(2x)$, the answer can also be written $h'(x) = 2 \cos(2x)$ — which is exactly what the chain rule will give next week.

Math 1231 — Quiz 2 (Version B)

Derivatives Sections 3.1–3.5

September 25, 2026 12 points

SOLUTIONS

Problem 1. The derivative as a limit.

(6 points)

(a) (2 points) Let $f(x) = |x + 3|$. Circle the statement that is true at $x = -3$.

- (A) f is differentiable at -3 , and $f'(-3) = 0$. (B) f is neither continuous nor differentiable at -3 .
(C) f is continuous at -3 but not differentiable at -3 . ← (D) f is differentiable at -3 , and $f'(-3) = 1$.

Solution. The graph of $y = |x + 3|$ is a V with its corner at $(-3, 0)$. There is no break there, so f is continuous: $\lim_{x \rightarrow -3} f(x) = 0 = f(-3)$.

Differentiability fails because the two one-sided difference quotients disagree. For $h > 0$, $|h| = h$; for $h < 0$, $|h| = -h$:

$$\lim_{h \rightarrow 0^+} \frac{|-3 + h + 3| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1, \quad \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1.$$

The one-sided limits are 1 and -1 , so the two-sided limit defining $f'(-3)$ does not exist. Answer: (C). This is the standard warning: differentiability implies continuity, but *continuity does not imply differentiability*. A corner is continuous and still has no tangent line.

(b) (2 points) Suppose f is differentiable on all of $\mathbb{R}$. Circle the correct value of

$$\lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3}.$$

- (A) $f(3)$ (B) $f'(3)$ ←
(C) $f'(x)$ (D) 0
(E) the slope of the secant line through $(3, f(3))$ and $(x, f(x))$ (F) the limit does not exist

Solution. No function is given and none is needed. This is the other form of the definition of the derivative at a point:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}, \quad \text{here with } a = 3.$$

Since f is differentiable everywhere, that limit exists and equals $f'(3)$. Answer: (B).

Why each wrong option is tempting:

- (A) $f(3)$ — the *value* of the function at 3, not the slope of its tangent there.
- (C) $f'(x)$ — the derivative *function*. The limit here is a single number: x is the variable being sent to 3, so it is gone once the limit is taken, and what remains depends only on the point 3.
- (D) 0 — the numerator does approach 0, but so does the denominator; $\frac{0}{0}$ is indeterminate, not 0.
- (E) — that is the quotient *before* the limit is taken. The secant slopes are what we take the limit of; the limit itself is the tangent slope.
- (F) — $\frac{0}{0}$ signals that a limit is waiting to be computed, not that it fails to exist. The hypothesis that f is differentiable at 3 guarantees it exists.

(c) (2 points) Let $f(x) = 5x + 1$. Find $f'(x)$ using the definition of the derivative as a limit. *No derivative rules.* Show each step.

Solution. Step 1. Build $f(x + h)$ by substituting $x + h$ everywhere x appears, then expand:

$$f(x + h) = 5(x + h) + 1 = 5x + 5h + 1.$$

Step 2. Form the difference quotient. The $5x$ and the $+1$ both cancel:

$$\frac{f(x + h) - f(x)}{h} = \frac{(5x + 5h + 1) - (5x + 1)}{h} = \frac{5h}{h}.$$

Step 3. Cancel the h . Substituting $h = 0$ before this step would give $\frac{0}{0}$; after it, nothing is left to worry about:

$$\frac{5h}{h} = 5.$$

Step 4. Take the limit of a constant:

$$f'(x) = \lim_{h \rightarrow 0} 5 = 5.$$

The difference quotient never depended on h at all, which is exactly what a *line* should do: $y = 5x + 1$ has slope 5 at every point.

Problem 2. Differentiation rules.

(6 points)

(a) (2 points) A particle's position at time t seconds is $s(t) = t^3 - 5t^2 + 7t$ meters. Circle its *velocity* at $t = -1$.

- (A) -13 m/s (B) 15 m/s (C) 20 m/s $\leftarrow$ (D) 0 m/s

Solution. Velocity is the derivative of position, $v(t) = s'(t)$. Differentiate term by term with the power rule:

$$s(t) = t^3 - 5t^2 + 7t \implies v(t) = s'(t) = 3t^2 - 10t + 7.$$

Now evaluate at $t = -1$, watching the signs: $(-1)^2 = 1$ and $-10(-1) = +10$.

$$v(-1) = 3(1) - 10(-1) + 7 = 3 + 10 + 7 = 20 \text{ m/s}.$$

Answer: **(C)**. (Choice (A) is $s(-1) = -1 - 5 - 7 = -13$, the particle's *position* rather than its velocity — the most common mistake here. Choice (B) comes from differentiating $-5t^2$ as $-5t$ instead of $-10t$, giving $3 + 5 + 7 = 15$.)

(b) (2 points) Let $f(x) = \frac{x^2}{x+2}$. Circle an equation of the line tangent to the graph of f at $x = 1$.

- (A) $y - \frac{1}{3} = \frac{5}{9}(x - 1) \leftarrow$ (B) $y - \frac{1}{3} = -\frac{5}{9}(x - 1)$
(C) $y - 1 = \frac{5}{9}\left(x - \frac{1}{3}\right)$ (D) $y - \frac{1}{3} = \frac{1}{3}(x - 1)$

Solution. A tangent line needs two things: a *point* and a *slope*.

The point. Evaluate f at $x = 1$:

$$f(1) = \frac{1^2}{1+2} = \frac{1}{3}, \quad \text{so the point is } \left(1, \frac{1}{3}\right).$$

The slope. This is a quotient. The quotient rule says: derivative of the top times the bottom, minus the top times the derivative of the bottom, all over the bottom squared. Here the top is x^2 and the bottom is $x + 2$:

$$f'(x) = \frac{(2x)(x+2) - (x^2)(1)}{(x+2)^2} = \frac{x^2 + 4x}{(x+2)^2}, \quad f'(1) = \frac{1+4}{3^2} = \frac{5}{9}.$$

Assemble. Substitute the point and the slope into point-slope form $y - y_0 = m(x - x_0)$:

$$y - \frac{1}{3} = \frac{5}{9}(x - 1).$$

Answer: **(A)**. (Simplifying to $y = \frac{5}{9}x - \frac{2}{9}$ is fine but not required.)

Why each wrong option is tempting:

- **(B)** the quotient-rule numerator written backwards — top times the derivative of the bottom *first* — giving slope $-\frac{5}{9}$. Order matters in the quotient rule; it does not in the product rule, which is why this slip is so common.
- **(C)** the coordinates of the point swapped — x_0 and y_0 traded places. The x -value 1 belongs with x , and $f(1) = \frac{1}{3}$ with y .
- **(D)** used the function value $f(1) = \frac{1}{3}$ as the *slope*. This line does pass through $(1, \frac{1}{3})$, so checking the point alone will not catch the error — the slope has to come from f' .

(c) (2 points) Let $h(x) = \cos(2x)$. The chain rule is not allowed. You may use the identity $\cos(2x) = 2\cos^2 x - 1$.

(i) Rewrite h using the identity, then find $h'(x)$ by the product rule.

(ii) Evaluate $h'(\frac{\pi}{4})$.

Solution. Step 1. The double-angle identity turns the composition into something built from products, which we *can* differentiate:

$$h(x) = \cos(2x) = 2\cos^2 x - 1 = 2(\cos x)(\cos x) - 1.$$

Step 2. Apply the product rule to $(\cos x)(\cos x)$, keeping the constant 2 out front; the constant -1 differentiates to 0:

$$h'(x) = 2\left[(-\sin x)(\cos x) + (-\sin x)(\cos x)\right] = -4\sin x \cos x.$$

Step 3. Evaluate at $x = \frac{\pi}{4}$, where $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$:

$$h'\left(\frac{\pi}{4}\right) = -4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -4 \cdot \frac{2}{4} = -2.$$

Remark. Since $2\sin x \cos x = \sin(2x)$, the answer can also be written $h'(x) = -2\sin(2x)$ — which is exactly what the chain rule will give next week.