Introduction to Limits

Sections 2.1 & 2.2

Calculus I — Lecture Notes

September 1, 2026

Calculus is what you get when algebra stops being good enough.

Last time we studied functions — the objects that change. Today we build the one tool that makes *all* of calculus run: the **limit**. Every big idea this semester (the derivative, the integral) is secretly a limit wearing a costume. The plan is to see *why* we need limits, then get a working, intuitive feel for what "lim" means, and how to read one off a table or a graph.

Two Problems That Started It All

Nearly all of calculus grows out of two very old, very concrete questions.

The tangent problem: how fast is a curve changing?

For a *line*, the rate of change is easy — it is the slope, one number that never changes. For a curve like $f(x) = x^2$ there is no single slope: the graph is nearly flat near the bottom and steepens as you move right. So we ask a sharper question: *what is the rate of change right at one point $x = a$?*

The trick is to sneak up on it. Pick the point $(a, f(a))$ we care about and a *second* nearby point $(x, f(x))$ on the curve. The straight line through those two points is called a **secant line**, and its slope we can compute with plain algebra.

Definition 1 (Secant line)**.** The **secant** to f through the points $(a, f(a))$ and $(x, f(x))$ is the line through those two points. Its slope is

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

Here is the whole idea in one sentence: **slide the second point x toward a, and watch the secant slopes settle down toward a single number.** That number is the slope of the **tangent line** at a — the true rate of change of f at that point. Sliding x toward a and asking what the slopes approach is exactly a *limit*.

Example 1 (Estimating a tangent slope with secant lines)**.** Estimate the slope of the tangent line to $f(x) = x^2$ at the point $(1, 1)$ by computing secant slopes through $(1, 1)$ and the points $(2, 4)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$.

Solution. Use $m_{\text{sec}} = \dfrac{f(x) - f(1)}{x - 1}$ with $f(1) = 1$.

$$\text{Through } (2,4): \quad m_{\text{sec}} = \frac{4-1}{2-1} = \frac{3}{1} = 3,$$

$$\text{Through } \left(\tfrac{3}{2}, \tfrac{9}{4}\right): \quad m_{\text{sec}} = \frac{\frac{9}{4} - 1}{\frac{3}{2} - 1} = \frac{\frac{5}{4}}{\frac{1}{2}} = \frac{5}{2} = 2.5.$$

The second point sits closer to $(1, 1)$, so its slope 2.5 is the better estimate. Push x even closer to 1 and the secant slopes keep sliding down — toward 2, as it happens. So the tangent slope at $(1, 1)$ is 2. We just computed a rate of change of a curve without any new machinery, only a limit-in-waiting.

Velocity is the same story in disguise. If $s(t)$ gives an object's position at time t, then over a time interval its **average velocity** is

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$$

— literally a secant slope. Letting t approach a turns average velocity into **instantaneous velocity**, the speed at a single instant. Slope of a tangent, speed at an instant: same limit, two costumes.

The area problem: how much has piled up?

The second question is: what is the area between the graph of a function and the x-axis over an interval $[a, b]$? For a curved top there is no area formula. So we approximate — slice the region into thin rectangles, add up their areas, then make the rectangles thinner and thinner. As the widths shrink toward zero, the rectangle sums approach the true area. That "approach a value as something shrinks to zero" is, once again, a limit. It is the seed of the **integral**.

Two different-looking problems, one common move: *approximate, then take a limit.* That is the whole reason we now stop and study limits carefully.

The Intuitive Idea of a Limit

Consider

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

This function is *undefined* at $x = 2$: plug in 2 and you get $\frac{0}{0}$, which is nonsense. But nothing stops us from asking what happens to $f(x)$ as x gets *close* to 2 without ever equaling 2. Factor the top:

$$f(x) = \frac{(x - 2)(x + 2)}{x - 2} = x + 2 \quad (\text{for } x \neq 2).$$

So near $x = 2$ the outputs hug $2 + 2 = 4$, even though $f(2)$ itself does not exist. We write this as

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4,$$

read "the limit of $f(x)$, as x approaches 2, equals 4."

Definition 2 (Intuitive definition of a limit)**.** Let $f(x)$ be defined for all x in an open interval around a, except possibly at a itself, and let L be a real number. If the values $f(x)$ get closer and closer to L — and stay close — as x gets closer and closer to a (from either side, with $x \neq a$), then we say the **limit** of $f(x)$ as x approaches a is L, and write

$$\lim_{x \to a} f(x) = L.$$

The single most important thing to notice: **the limit is about what happens *near* a, not *at* a.** The value $f(a)$ can be missing, or it can even be some other number, and the limit does not care. Limits look at the neighborhood, not the house.

Estimating Limits from Tables

When we cannot factor our way to the answer, we estimate. Build a table: feed in x-values creeping toward a from the left ($x < a$) and from the right ($x > a$), and watch where the outputs head.

Example 2 (Estimating a limit from a table)**.** Estimate $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$ using a table.

Solution. At $x = 4$ the formula gives $\frac{0}{0}$, undefined — so we approach 4 from both sides and tabulate.

x	$\dfrac{\sqrt{x}-2}{x-4}$
3.9	$0.25158\ldots$
3.99	$0.25016\ldots$
3.999	$0.25002\ldots$
4.001	$0.24998\ldots$
4.01	$0.24984\ldots$
4.1	$0.24846\ldots$

Coming from the left the values *rise* toward 0.25; coming from the right they *fall* toward 0.25. Both sides agree, so

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = 0.25 = \frac{1}{4}.$$

(A table gives strong evidence, not a proof — but for now it is exactly how we build intuition.)

Tables can also warn you when a limit **does not exist**. If the left-hand values and right-hand values head toward *different* numbers, or if the outputs never settle at all, there is no single L, and we say the limit **does not exist** (abbreviated **DNE**).

Estimating Limits from Graphs

Often the fastest way to read a limit is to look at a picture. To find $\lim_{x\to a} f(x)$, trace the graph toward $x = a$ from both sides and ask: what height is the curve heading for? Ignore whatever is happening *at* a itself — an open hole, a filled dot in the wrong place, none of it changes the limit.

Example 3 (Reading a limit off a graph)**.** A function g is drawn as a smooth increasing curve, except that at $x = -1$ the curve passes through height 3 with a small *open circle*, while a separate *filled dot* is placed at height 4. So $g(-1) = 4$, but the curve itself approaches height 3. Find $\lim_{x\to -1} g(x)$.

Solution. As x approaches -1 from the left and from the right, the curve heads for height 3 from both directions. The stray point at 4 is the *value* $g(-1)$, and the limit ignores it. Therefore

$$\lim_{x\to -1} g(x) = 3, \qquad \text{even though } g(-1) = 4.$$

This is the definition doing its job: a limit can exist, the function can be defined there, and the two can simply disagree.

One-Sided Limits

Sometimes a function behaves differently depending on which direction you come from. For that we need **one-sided limits**.

Definition 3 (One-sided limits)**.** The **limit from the left**,

$$\lim_{x\to a^-} f(x) = L,$$

means $f(x)$ approaches L as x approaches a through values *less* than a. The **limit from the right**,

$$\lim_{x\to a^+} f(x) = L,$$

means $f(x)$ approaches L as x approaches a through values *greater* than a.

The two-sided limit is just the two one-sided limits forced to agree.

Definition 4 (Relating one-sided and two-sided limits)**.** $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$. If the left and right limits disagree, then $\lim_{x\to a} f(x)$ does not exist.

Example 4 (One-sided limits of a piecewise function)**.** For

$$f(x) = \begin{cases} x+1, & x < 2, \\ x^2 - 4, & x \geq 2, \end{cases}$$

find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $\lim_{x\to 2} f(x)$.

Solution. Approaching from the left we are below 2, so the rule $x+1$ applies. A short table ($x = 1.9,\ 1.99,\ 1.999$) gives outputs $2.9,\ 2.99,\ 2.999$, heading to 3:

$$\lim_{x\to 2^-} f(x) = 2 + 1 = 3.$$

Approaching from the right we are at or above 2, so the rule $x^2 - 4$ applies. A table ($x = 2.1,\ 2.01,\ 2.001$) gives $0.41,\ 0.0401,\ 0.004001$, heading to 0:

$$\lim_{x\to 2^+} f(x) = 2^2 - 4 = 0.$$

The two sides give 3 and 0 — they disagree. Therefore

$$\lim_{x\to 2} f(x) \text{ does not exist (DNE).}$$

The graph has a *jump* at $x = 2$, and a jump is exactly what a disagreeing pair of one-sided limits looks like.

Practice (work this out). Use a table to estimate $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$ by trying $x = 0.9,\ 0.99,\ 0.999$ and $x = 1.1,\ 1.01,\ 1.001$. Does the limit exist, and if so, what is it? (Hint: the function is undefined at $x = 1$; try factoring the top to check your table.)

Infinite Limits and Vertical Asymptotes

There is one more way a limit can fail to be an ordinary number: the outputs can blow up. Consider

$$h(x) = \frac{1}{(x-2)^2}.$$

As x nears 2, the denominator $(x-2)^2$ is tiny and positive, so $h(x)$ grows without bound. The values don't approach any real number — they increase forever. We describe this with the symbol $+\infty$ and write

$$\lim_{x\to 2} \frac{1}{(x-2)^2} = +\infty.$$

Definition 5 (Infinite limit)**.** If $f(x)$ increases without bound as x approaches a, we write $\lim\limits_{x\to a} f(x) = +\infty$; if $f(x)$ decreases without bound, we write $\lim\limits_{x\to a} f(x) = -\infty$. These are **infinite limits**. The same idea applies one-sidedly, giving $\lim_{x\to a^-}$ and $\lim_{x\to a^+}$.

A warning about words: when we write $\lim = +\infty$ we are *not* saying the limit "exists" as a number. The symbol $+\infty$ is a description of runaway behavior, nothing more. It is more informative than plain DNE, so we prefer it when it applies.

Example 5 (A one-sided infinite limit and a vertical asymptote)**.** Investigate $\lim\limits_{x\to 0^-} \dfrac{1}{x}$, $\lim\limits_{x\to 0^+} \dfrac{1}{x}$, and $\lim\limits_{x\to 0} \dfrac{1}{x}$.

Solution. Tabulate near 0 from each side.

x	$\frac{1}{x}$
-0.1	-10
-0.01	-100
-0.001	-1000
0.001	1000
0.01	100
0.1	10

From the left, $\frac{1}{x}$ plunges without bound: $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. From the right it climbs without bound: $\lim_{x\to 0^+} \frac{1}{x} = +\infty$. The two sides run off in opposite directions, so the two-sided limit

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$

Because the function shoots off to $\pm\infty$ at $x = 0$, the vertical line $x = 0$ is a **vertical asymptote** of the graph.

Definition 6 (Vertical asymptote)**.** The line $x = a$ is a **vertical asymptote** of f if at least one of the one-sided limits $\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)$ is $+\infty$ or $-\infty$.

Wherever a limit runs off to $\pm\infty$, you have found a vertical asymptote — a line the graph hugs but never crosses. For a function like $\dfrac{1}{(x-a)^n}$, the point $x = a$ is always a vertical asymptote.

Practice (work this out). For $f(x) = \dfrac{1}{(x+3)^4}$, decide whether $\lim\limits_{x\to -3^-} f(x)$ and $\lim\limits_{x\to -3^+} f(x)$ are $+\infty$ or $-\infty$, state $\lim\limits_{x\to -3} f(x)$, and name the vertical asymptote. (Hint: an even power keeps the denominator positive on both sides.)

Summary

- Calculus starts from two problems — the **tangent problem** (rate of change of a curve) and the **area problem** (area under a curve). Both are solved by *approximate, then take a limit.*
- The tangent slope at a is the limit of **secant slopes** $\dfrac{f(x) - f(a)}{x - a}$ as $x \to a$; instantaneous velocity is the same limit applied to position.
- $\lim_{x \to a} f(x) = L$ means $f(x)$ stays close to L as x nears a. The limit depends on behavior *near* a, never on $f(a)$ itself.
- Estimate limits from a **table** (approach from both sides) or a **graph** (trace toward a, ignore the point at a).
- **One-sided limits** (a^-, a^+) must agree for the two-sided limit to exist; a disagreement is a *jump*, and the limit is DNE.
- **Infinite limits** ($\pm\infty$) describe outputs that blow up; each one marks a **vertical asymptote** $x = a$.
- Next time: the **Limit Laws** — how to compute limits by algebra instead of tables.