The Precise Definition of a Limit

Section 2.5

Calculus I — Lecture Notes

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From Intuition to Precision

"Close" is not a feeling — it is a challenge, and we can always win it.

So far we have said $\lim_{x\to a} f(x) = L$ means that $f(x)$ gets "close to" L as x gets "close to" a. That intuition has carried us a long way, but it hides a real question: *how close is close?* If a skeptic demands that $f(x)$ land within one one-thousandth of L, can we guarantee it? Within one millionth? Within any tolerance at all?

Today we replace the vague words with a precise, testable definition — the famous **epsilon-delta definition**. It is widely regarded as one of the hardest definitions in a first calculus course, but the idea underneath is a simple game: you name a tolerance, and I must find a window of inputs that keeps me inside it. If I can always answer your challenge, the limit is L.

Quantifying Closeness

Everything rests on one fact: the distance between two numbers a and b on the number line is $|a-b|$. So "$f(x)$ is close to L" really means $|f(x)-L|$ is small, and "x is close to a" means $|x-a|$ is small. Two translations we will use constantly:

- $|f(x)-L| < \varepsilon$ means the distance from $f(x)$ to L is less than ε; equivalently $L-\varepsilon < f(x) < L+\varepsilon$.
- $0 < |x-a| < \delta$ means x is within δ of a *but not equal to* a; equivalently $a-\delta < x < a+\delta$ with $x \neq a$.

The "$0 <$" part matters: the limit is about what happens *near* a, never at a itself. Here ε (epsilon) is the tolerance on the *output* and δ (delta) is the width of the window on the *input*.

Definition 1 (The Epsilon-Delta Definition of a Limit)**.** Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a, and let L be a real number. Then

$$\lim_{x\to a} f(x) = L$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$.

Read it as a challenge and a response. The word *every* says the skeptic may pick ε as small as they like. The words *there exists* say it is our job to produce a δ that works. The closer we are told to be (smaller ε), the narrower our input window must be (smaller δ). Notice δ is allowed to depend on ε — in fact it almost always does.

The Geometric Meaning

Picture two horizontal lines at heights $L - \varepsilon$ and $L + \varepsilon$; they cut out a horizontal band of allowed outputs. The skeptic makes this band as thin as they wish. Our task: find two vertical lines at $x = a - \delta$ and $x = a + \delta$ so that, for every input in that vertical strip (except possibly $x = a$), the graph stays inside the horizontal band.

If the limit really is L, then no matter how thin the band gets, we can always shrink the strip enough to trap the graph. If the graph "escapes" the band no matter how narrow we make the strip, the limit is not L. This picture is the whole definition, drawn.

Proving a Linear Limit

The cleanest proofs are for lines, because the distance $|f(x) - L|$ is a constant multiple of $|x - a|$, so solving for δ is just division.

Example 1 (A linear proof)**.** Prove that $\lim_{x \to 1}(2x + 1) = 3$.

Solution. We work backward from what we want, then write it forward.

Scratch work (finding δ). We want $|(2x + 1) - 3| < \varepsilon$. Simplify the left side to connect it to $|x - 1|$:

$$|(2x + 1) - 3| = |2x - 2| = 2\,|x - 1|.$$

So $|(2x + 1) - 3| < \varepsilon$ is the same as $2\,|x - 1| < \varepsilon$, i.e. $|x - 1| < \varepsilon/2$. That tells us exactly what window to use: choose $\delta = \varepsilon/2$.

The proof. Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{2}$. Assume $0 < |x - 1| < \delta$. Then

$$|(2x + 1) - 3| = |2x - 2| = 2\,|x - 1| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\lim_{x \to 1}(2x + 1) = 3$. ■

The pattern is worth naming, because every one of these proofs follows it.

Strategy: proving $\lim_{x \to a} f(x) = L$.

1. Begin: "Let $\varepsilon > 0$."
2. Do scratch work on $|f(x) - L| < \varepsilon$ to solve for how small $|x - a|$ must be; this reveals δ. Then write: "Choose $\delta =$ ___."
3. State: "Assume $0 < |x - a| < \delta$."
4. Show the chain $|f(x) - L| = \cdots < \varepsilon$, using the assumption and your choice of δ.
5. Conclude: "Therefore $\lim_{x \to a} f(x) = L$."

Example 2 (A linear proof with a negative slope and limit)**.** Prove that $\lim_{x \to -1}(4x + 1) = -3$.

Solution. Scratch work. We want $|(4x + 1) - (-3)| < \varepsilon$. Simplify:

$$|(4x + 1) + 3| = |4x + 4| = 4\,|x + 1| = 4\,|x - (-1)|.$$

So we need $4\,|x - (-1)| < \varepsilon$, i.e. $|x - (-1)| < \varepsilon/4$. Choose $\delta = \varepsilon/4$.

The proof. Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{4}$. Assume $0 < |x - (-1)| < \delta$. Then

$$|(4x + 1) - (-3)| = |4x + 4| = 4\,|x + 1| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

Therefore $\lim_{x \to -1}(4x + 1) = -3$. ■

Practice (work this out). Use the epsilon-delta definition to prove that $\lim_{x\to 2}(3x-2)=4$. (Find δ in terms of ε, then write the proof.)

When the Function Is Not Linear

For a curve, $|f(x)-L|$ is no longer a constant times $|x-a|$, so we cannot just divide. Two things change: (1) we often first agree to only look at x within some fixed distance of a (say $\delta \le 1$), which lets us bound the "extra" factor, and (2) our final δ becomes the *smaller* of two constraints, written with min.

Example 3 (A quadratic, the geometric way)**.** Prove that $\lim_{x\to 2} x^2 = 4$.

Solution. Let $\varepsilon > 0$. We may as well assume $\varepsilon \le 4$ (if we can win the challenge for small tolerances, the large ones are automatic).

We want $|x^2-4|<\varepsilon$, i.e. $4-\varepsilon < x^2 < 4+\varepsilon$. Since x is near 2, it is positive, so take square roots:

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}.$$

This is a window around 2, but it is *lopsided*: the distance from 2 down to $\sqrt{4-\varepsilon}$ differs from the distance from 2 up to $\sqrt{4+\varepsilon}$. To fit a symmetric window $(2-\delta, 2+\delta)$ inside it, take the smaller of the two distances:

$$\delta = \min\left\{\, 2-\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}-2 \,\right\}.$$

With this δ, assuming $0<|x-2|<\delta$ gives $\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$; squaring (all parts positive) gives $4-\varepsilon < x^2 < 4+\varepsilon$, which is exactly $|x^2-4|<\varepsilon$. Therefore $\lim_{x\to 2} x^2 = 4$. ■

The geometric min works, but it is clumsy. The algebraic method below is the one you will actually reuse, because it handles any quadratic.

Example 4 (A quadratic, the algebraic way)**.** Prove that $\lim_{x\to -1}(x^2-2x+3)=6$.

Solution. Scratch work. We want $|(x^2-2x+3)-6|<\varepsilon$. Factor the inside:

$$(x^2-2x+3)-6 = x^2-2x-3 = (x+1)(x-3),$$

so $|(x^2-2x+3)-6| = |x+1|\cdot|x-3|$. The factor $|x+1| = |x-(-1)|$ is what δ controls directly. The other factor $|x-3|$ is a nuisance — we must bound it by a number, because δ may only depend on ε.

Taming $|x-3|$. First insist $\delta \le 1$. Then $|x+1|<1$ means $-1<x+1<1$, so $-2<x<0$, so $-5<x-3<-3$, giving $|x-3|<5$. Now

$$|x+1|\cdot|x-3| < |x+1|\cdot 5,$$

and we can force this below ε by also insisting $|x+1|<\varepsilon/5$. Both requirements hold at once if we take $\delta = \min\{1, \varepsilon/5\}$.

The proof. Let $\varepsilon>0$. Choose $\delta=\min\{1,\varepsilon/5\}$. Assume $0<|x+1|<\delta$. Since $\delta\le 1$ we have $|x+1|<1$, hence $-5<x-3<-3$ and so $|x-3|<5$. Since also $|x+1|<\varepsilon/5$,

$$\left|(x^2-2x+3)-6\right| = |x+1|\cdot|x-3| < \frac{\varepsilon}{5}\cdot 5 = \varepsilon.$$

Therefore $\lim_{x\to -1}(x^2-2x+3)=6$. ■

Practice (work this out). Use the epsilon-delta definition to prove that $\lim_{x\to 3}(x^2-9)=0$. (Factor the difference, restrict $\delta \le 1$ to bound the loose factor, then choose $\delta = \min\{1, \cdots\}$.)

One-Sided and Infinite Limits

The same machine, lightly modified, defines one-sided and infinite limits.

Definition 2 (One-sided limits)**.** $\lim_{x\to a^+} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$. Likewise $\lim_{x\to a^-} f(x) = L$ if the same holds whenever $-\delta < x - a < 0$.

The only change is the input condition: $0 < x - a < \delta$ looks only to the *right* of a, and $-\delta < x - a < 0$ only to the *left*.

Definition 3 (Infinite limit)**.** $\lim_{x\to a} f(x) = +\infty$ if for every $M > 0$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$.

Here the output challenge flips: instead of "get within ε of L," the challenge is "get above every height M." We answer, as always, with a δ.

Summary

- Distance is absolute value: $|f(x) - L| < \varepsilon$ means $f(x)$ is within ε of L; $0 < |x - a| < \delta$ means x is within δ of a but not equal to a.
- **Definition:** $\lim_{x\to a} f(x) = L$ means for every $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x - a| < \delta$ forces $|f(x) - L| < \varepsilon$. Think challenge (ε) and response (δ).
- **Geometry:** whatever horizontal ε-band the skeptic draws around L, we can find a vertical δ-strip around a that traps the graph inside the band.
- **Linear proofs:** $|f(x)-L|$ is a constant times $|x-a|$, so divide to get δ directly (e.g. $\delta = \varepsilon/2$).
- **Quadratic proofs:** factor $|f(x) - L| = |x - a| \cdot$ (loose factor), restrict $\delta \le 1$ to bound the loose factor by a number, then take $\delta = \min\{1, \varepsilon/(\text{that number})\}$.
- **Variations:** one-sided limits change only the input condition; infinite limits replace the ε-target with "exceed every height M."