

Introduction to Derivatives

Sections 3.1 & 3.2

Calculus I — Lecture Notes

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The whole subject in one idea: zoom in on a curve until it looks like a line, and measure that line's slope.

We spent the last few classes building the **limit**, and it may have felt like machinery without a job. Today it gets its job. Every question about how fast something is changing — the speed of a falling ball, the rate a profit grows, how steeply a road climbs — is answered by the same construction: the slope of a line that touches a curve at a single point. That number is called the **derivative**, and building it is the entire content of today's lecture. We start with slopes of lines you already know, take a limit, and land on the central object of the course.

From Secant Lines to the Tangent Line

Given a curve $y = f(x)$ and a point $(a, f(a))$ on it, we want the slope of the curve *at that single point*. But slope is a thing that needs two points — you cannot compute $\frac{\Delta y}{\Delta x}$ from one point alone. So we cheat, temporarily: pick a *second* nearby point $(x, f(x))$ and draw the line through both. That line is a **secant line**, and its slope is a **difference quotient**:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

There is a second way to name the second point. Instead of calling it x , call it $a + h$, where h is the small *increment* from a . Then the same secant slope reads

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}.$$

These two forms say the same thing; we pick whichever is easier to compute.

Now the key move. As the second point slides toward $(a, f(a))$ — that is, as $x \rightarrow a$, equivalently $h \rightarrow 0$ — the secant lines pivot and settle onto one limiting line that just grazes the curve at a . That limiting line is the **tangent line**, and its slope is the limit of the secant slopes.

Definition 1 (Tangent line). Let f be defined on an open interval containing a . The **tangent line** to $y = f(x)$ at a is the line through $(a, f(a))$ whose slope is

$$m_{\text{tan}} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h},$$

provided the limit exists. Both forms give the same number.

Notice *why* we needed limits so badly. Plugging $x = a$ straight into the difference quotient gives $\frac{0}{0}$ — undefined. The limit is exactly the tool that lets us push the two points together *without* literally dividing by zero. That $\frac{0}{0}$ is not a dead end; it is a signal that a limit is waiting to be computed.

Example 1 (Finding a tangent line). Find an equation of the line tangent to $f(x) = x^2$ at $x = 3$.

Solution. First get the slope from the definition, using the $x \rightarrow a$ form:

$$\begin{aligned} m_{\text{tan}} &= \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3} (x + 3) = 6. \end{aligned}$$

The factor $x - 3$ is what cancels the troublesome zero in the denominator. Now find a point: since the line is tangent at $x = 3$, it passes through $(3, f(3)) = (3, 9)$. Point-slope form gives $y - 9 = 6(x - 3)$, i.e.

$$y = 6x - 9.$$

Example 2 (Same slope, other form). Recompute the slope of the tangent to $f(x) = x^2$ at $x = 3$ using the $h \rightarrow 0$ form.

Solution.

$$\begin{aligned} m_{\text{tan}} &= \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h} \\ &= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \rightarrow 0} \frac{h(6 + h)}{h} = \lim_{h \rightarrow 0} (6 + h) = 6. \end{aligned}$$

Same answer, 6. The two definitions are interchangeable; here the algebra hinged on expanding $(3 + h)^2$ and then cancelling the common factor h .

Example 3 (A tangent to a rational function). Find an equation of the line tangent to $f(x) = \frac{1}{x}$ at $x = 2$.

Solution. Use the $x \rightarrow a$ form. The subtraction of fractions is the only wrinkle:

$$\begin{aligned} m_{\text{tan}} &= \lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = \lim_{x \rightarrow 2} \frac{\frac{2-x}{2x}}{x - 2} = \lim_{x \rightarrow 2} \frac{2 - x}{2x(x - 2)} \\ &= \lim_{x \rightarrow 2} \frac{-(x - 2)}{2x(x - 2)} = \lim_{x \rightarrow 2} \frac{-1}{2x} = -\frac{1}{4}. \end{aligned}$$

The line passes through $(2, f(2)) = (2, \frac{1}{2})$, so point-slope gives $y - \frac{1}{2} = -\frac{1}{4}(x - 2)$, which simplifies to

$$y = -\frac{1}{4}x + 1.$$

Practice (work this out). Find the slope of the line tangent to $f(x) = x^2 + x$ at $x = 1$, then write an equation for that tangent line.

The Derivative at a Point

That limit-of-slopes shows up so relentlessly — in velocity, in growth rates, in marginal cost — that it earns its own name and symbol. The process of computing it is called **differentiation**.

Definition 2 (Derivative at a point). Let f be defined on an open interval containing a . The **derivative of f at a** , written $f'(a)$, is

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h},$$

provided this limit exists. Geometrically, $f'(a)$ is the slope of the tangent line to $y = f(x)$ at a .

So “the derivative at a ” and “the slope of the tangent at a ” are two names for one number. The derivative also has a physical name: it is the **instantaneous rate of change** of f at a . If $s(t)$ is the position of a moving object, then $\frac{s(t)-s(a)}{t-a}$ is its *average* velocity over $[a, t]$, and the limit $s'(a)$ is its *instantaneous* velocity at the instant a — the number your speedometer reads.

Example 4 (Computing a derivative from the definition). For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$.

Solution. First note $f(2) = 3(4) - 8 + 1 = 5$. Use the $x \rightarrow a$ form:

$$\begin{aligned} f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{(3x^2 - 4x + 1) - 5}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{3x^2 - 4x - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(3x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (3x + 2) = 8. \end{aligned}$$

The rate of change of f at $x = 2$ is 8.

Example 5 (Estimating a derivative from a table). For $f(x) = x^2$, estimate $f'(3)$ numerically.

Solution. We cannot plug $x = 3$ into $\frac{x^2-9}{x-3}$ directly, but we can watch the difference quotient as x closes in on 3 from both sides.

x	$\frac{x^2 - 9}{x - 3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

The values crowd in on 6 from both sides, so $f'(3) \approx 6$ — matching the exact answer we found earlier. Tables are how we estimate a derivative when the algebra is hard or when f is only given by data.

Practice (work this out). For $f(x) = x^2 + 3x + 2$, find $f'(1)$ using the definition.

The Derivative as a Function

Computing $f'(a)$ one input at a time is tedious. But nothing forced us to pick a specific number for a — we can leave it as the variable x and get a *formula* that returns the slope at every input at once. That formula is a new function, the **derivative function**.

Definition 3 (Derivative function). The **derivative** of f is the function f' defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

whose domain is all x for which this limit exists.

Example 6 (Derivative of a quadratic). Find $f'(x)$ for $f(x) = x^2 - 2x$.

Solution. Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ into the definition:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 - 2(x+h)] - [x^2 - 2x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h - 2)}{h} \\ &= \lim_{h \rightarrow 0} (2x + h - 2) = 2x - 2. \end{aligned}$$

Every term without a factor of h had to cancel before we could take the limit — that is the mechanism of the whole computation. Now f' is a formula: the slope at $x = 1$ is $f'(1) = 0$ (a horizontal tangent), at $x = 3$ it is $f'(3) = 4$, and so on.

Example 7 (Derivative of a square-root function). Find $f'(x)$ for $f(x) = \sqrt{x}$.

Solution. The difference quotient has a root on top, so we multiply by the conjugate to move the trouble downstairs:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

Two things to read off. First, $f'(x) = \frac{1}{2\sqrt{x}} > 0$ everywhere it is defined — consistent with $\sqrt{x}$ always rising. Second, f' is undefined at $x = 0$ and blows up as $x \rightarrow 0^+$; the curve has a *vertical tangent* there, a case we return to below.

Notation

There are several standard symbols for the derivative of $y = f(x)$, all meaning the same thing:

$$f'(x), \quad y', \quad \frac{dy}{dx}, \quad \frac{d}{dx}(f(x)).$$

The symbol $\frac{dy}{dx}$ is **Leibniz notation**. It is not a fraction, but it remembers where the derivative came from: the slope $\frac{\Delta y}{\Delta x}$ of a secant, in the limit as $\Delta x \rightarrow 0$,

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

To denote the derivative *at a point* in this notation we write $\left. \frac{dy}{dx} \right|_{x=a}$, which means the same as $f'(a)$. The forms coexist because different settings make different ones convenient — f' is compact, while $\frac{dy}{dx}$ keeps the variables visible, which matters in physics and later in the chain rule.

Practice (work this out). Use the definition to find $f'(x)$ for $f(x) = 5x - x^2$.

Differentiability and Continuity

A function is **differentiable at a** if $f'(a)$ exists, and **differentiable on a set** if it is differentiable at every point of that set. How does this relate to continuity? One direction is guaranteed.

Definition 4 (Differentiability implies continuity). If f is differentiable at a , then f is continuous at a .

The idea of the proof: for the difference quotient $\frac{f(x)-f(a)}{x-a}$ to approach a *finite* slope as $x \rightarrow a$, the numerator $f(x) - f(a)$ has no choice but to approach 0 — otherwise the fraction would explode. And $f(x) \rightarrow f(a)$ is exactly what continuity at a means. A curve with a well-defined tangent line cannot have a break in it.

The converse is false. Continuity does *not* imply differentiability. A function can be perfectly connected — no breaks — and still have no well-defined slope at a point. There are three classic ways this happens.

1. Corners

The absolute value $f(x) = |x|$ is continuous everywhere, but its graph has a sharp corner at the origin. Check the derivative there from the definition:

$$f'(0) = \lim_{x \rightarrow 0} \frac{|x| - 0}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x}.$$

Approaching from each side gives a different answer:

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1 \quad \text{but} \quad \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1.$$

The two one-sided limits disagree, so the limit does not exist and $f'(0)$ is undefined. Geometrically, the slope coming in from the left (-1) and from the right ($+1$) never agree — there is no single tangent line at a corner.

2. Cusps and vertical tangents

Consider $f(x) = \sqrt[3]{x}$, the cube root. It is continuous everywhere, but at the origin

$$f'(0) = \lim_{x \rightarrow 0} \frac{\sqrt[3]{x} - 0}{x - 0} = \lim_{x \rightarrow 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

The limit is infinite, not a finite number, so $f'(0)$ does not exist. The graph has a **vertical tangent** at 0: the curve is smooth, but at that instant it is rising infinitely steeply, and a vertical line has no slope. (We saw the same behavior for $\sqrt{x}$ at 0 a moment ago.) A **cusp**, where the curve comes to a point with the slope racing to $+\infty$ on one side and $-\infty$ on the other, fails to be differentiable for the same reason.

3. Discontinuities

Finally, wherever f is *not* continuous — a jump, a hole, a break — it cannot be differentiable either, since we just argued differentiability forces continuity. A broken graph has no tangent line at the break.

Example 8 (Where is this function not differentiable?). Where does $f(x) = |x - 2|$ fail to be differentiable, and why?

Solution. Rewrite it piecewise:

$$f(x) = \begin{cases} -(x - 2), & x < 2, \\ (x - 2), & x \geq 2. \end{cases}$$

For $x < 2$ the graph is a line of slope -1 ; for $x > 2$ it is a line of slope $+1$. Everywhere except $x = 2$ the function is a straight line, hence differentiable. At $x = 2$ the left slope -1 and right slope $+1$ disagree, so this is a corner and $f'(2)$ does not exist. The function is continuous everywhere but differentiable everywhere *except* $x = 2$.

Practice (work this out). The function $f(x) = x^{2/3}$ is continuous at $x = 0$. Set up the limit $\lim_{x \rightarrow 0} \frac{x^{2/3} - 0}{x - 0}$ and use it to explain why f is not differentiable there. What feature does the graph have at the origin?

Summary

- The slope of a curve at a point is defined as a limit of **secant** slopes (difference quotients); the limiting line is the **tangent line**.
- Two interchangeable forms: $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$.
- The **derivative at** a , $f'(a)$, is the slope of the tangent at a and the **instantaneous rate of change** of f (e.g. velocity).
- Leaving a as a variable gives the **derivative function** $f'(x)$; compute it once, read the slope anywhere. Notation: $f'(x)$, y' , $\frac{dy}{dx}$, $\frac{d}{dx}f(x)$.
- **Differentiable** $\Rightarrow$ **continuous**, but *not* conversely.
- A continuous function fails to be differentiable at **corners** (left and right slopes disagree, e.g. $|x|$), at **cusps** / **vertical tangents** (slope $\rightarrow \pm\infty$, e.g. $\sqrt[3]{x}$), and at any **discontinuity**.
- Next time: rules that compute derivatives without the limit every time.