

# Differentiation Rules

## Section 3.3

### Calculus I — Lecture Notes

September 17, 2026

## Why We Need Rules

*We already have the definition of the derivative. Today we stop using it.*

Computing a derivative straight from the limit definition

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

works, but it is slow, and for something like  $f(x) = 2x^5 + 7$  it is genuinely painful. The good news: every derivative you will ever need for a polynomial or a rational function is built from a small handful of rules. Learn these six rules and you can differentiate almost anything by inspection — no limits required. We derive each rule once (from the definition, so you trust it), then use it forever after.

## The Constant and Power Rules

The two functions  $f(x) = c$  (a constant) and  $g(x) = x^n$  are the building blocks of every polynomial. We start there.

**Definition 1** (The Constant Rule). If  $f(x) = c$  is constant, then  $f'(x) = 0$ ; equivalently

$$\frac{d}{dx}(c) = 0.$$

Why? A constant function is a horizontal line. Its slope — its rate of change — is 0 everywhere. The limit definition agrees:  $f(x+h) - f(x) = c - c = 0$ , so the difference quotient is  $0/h = 0$  for every  $h$ , and the limit is 0.

**Definition 2** (The Power Rule). If  $n$  is a positive integer and  $f(x) = x^n$ , then

$$f'(x) = n x^{n-1}; \quad \text{equivalently} \quad \frac{d}{dx} x^n = n x^{n-1}.$$

The pattern in words: *the power drops down in front as a coefficient, and the new power is one smaller.* We saw earlier that  $\frac{d}{dx}(x^2) = 2x$ ; the power rule says the same about every power at once. (It in fact holds for negative integer powers too, which we exploit at the end of the lecture.) Watch out for one trap: the rule is for a *variable base raised to a constant power*, like  $x^n$ .

**Example 1** (Applying the constant and power rules). Find each derivative:  $\frac{d}{dx}(8)$ ,  $\frac{d}{dx}(x^{10})$ ,  $\frac{d}{dx}(x^7)$ .

*Solution.* The first is a constant, so its derivative is 0. For the other two, drop the power in front and subtract one:

$$\frac{d}{dx}(8) = 0, \quad \frac{d}{dx}(x^{10}) = 10x^9, \quad \frac{d}{dx}(x^7) = 7x^6.$$

## The Constant Multiple, Sum, and Difference Rules

Real functions are sums of scaled powers. Three more rules let us differentiate them term by term.

**Definition 3** (Constant Multiple, Sum, and Difference Rules). Let  $f$  and  $g$  be differentiable and let  $k$  be a constant. Then

$$\begin{aligned}\frac{d}{dx}(kf(x)) &= kf'(x), \\ \frac{d}{dx}(f(x) + g(x)) &= f'(x) + g'(x), \quad \frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x).\end{aligned}$$

In plain terms: a constant factor rides along untouched, and the derivative of a sum (or difference) is the sum (or difference) of the derivatives. These follow from the matching limit laws — the limit of a sum is the sum of the limits, and a constant pulls outside a limit — applied to the difference quotient. Together with the power rule, they let you differentiate any polynomial on sight.

**Example 2** (The constant multiple rule). Find the derivative of  $g(x) = 3x^2$ .

*Solution.* The constant 3 stays out front while we differentiate  $x^2$ :

$$g'(x) = 3 \frac{d}{dx}(x^2) = 3(2x) = 6x.$$

Since  $f(x) = x^2$  has derivative  $2x$ , the derivative of  $g$  is exactly 3 times the derivative of  $f$  — scaling the function scales its slope.

**Example 3** (A full polynomial). Find the derivative of  $f(x) = 2x^5 + 7$ .

*Solution.* Split across the sum, then handle each piece.

$$\begin{aligned}f'(x) &= \frac{d}{dx}(2x^5) + \frac{d}{dx}(7) && \text{sum rule} \\ &= 2 \frac{d}{dx}(x^5) + 0 && \text{constant multiple and constant rules} \\ &= 2(5x^4) && \text{power rule} \\ &= 10x^4. && \text{simplify}\end{aligned}$$

**Example 4** (Equation of a tangent line). Find an equation of the line tangent to  $f(x) = x^2 - 4x + 6$  at  $x = 1$ .

*Solution.* A tangent line needs a point and a slope. The point:

$$f(1) = 1^2 - 4(1) + 6 = 3, \quad \text{so the point is } (1, 3).$$

The slope is  $f'(1)$ , so differentiate term by term:

$$f'(x) = 2x - 4, \quad f'(1) = 2(1) - 4 = -2.$$

Point-slope form through  $(1, 3)$  with slope  $-2$ :

$$y - 3 = -2(x - 1) \implies y = -2x + 5.$$

**Practice (work this out).** Find the derivative of  $f(x) = 2x^3 - 6x^2 + 3$ .

## The Product Rule

It is tempting to guess that the derivative of a product is the product of the derivatives. It is not. Quick check: for  $f(x) = x^2 = x \cdot x$  the true derivative is  $2x$ , but multiplying the derivatives of the two factors gives  $1 \cdot 1 = 1$ . The correct rule mixes the pieces.

**Definition 4** (The Product Rule). If  $f$  and  $g$  are differentiable, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x).$$

In words: the derivative of the first times the second, plus the derivative of the second times the first.

**Example 5** (Product rule from given values). Let  $j(x) = f(x)g(x)$ . Find  $j'(2)$  given  $f(2) = 3$ ,  $f'(2) = -4$ ,  $g(2) = 1$ , and  $g'(2) = 6$ .

*Solution.* Apply the rule, then plug in the numbers:

$$j'(x) = f'(x)g(x) + g'(x)f(x),$$

$$j'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = -4 + 18 = 14.$$

**Example 6** (Product of two polynomials). For  $j(x) = (x^2 + 2)(3x^3 - 5x)$ , find  $j'(x)$  with the product rule, then check by multiplying out first.

*Solution.* Take  $f(x) = x^2 + 2$  and  $g(x) = 3x^3 - 5x$ , so  $f'(x) = 2x$  and  $g'(x) = 9x^2 - 5$ . Then

$$j'(x) = f'(x)g(x) + g'(x)f(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2).$$

Expanding and collecting like terms:

$$j'(x) = (6x^4 - 10x^2) + (9x^4 + 18x^2 - 5x^2 - 10) = 15x^4 + 3x^2 - 10.$$

Check: multiplying first gives  $j(x) = 3x^5 + x^3 - 10x$ , and differentiating that directly gives  $j'(x) = 15x^4 + 3x^2 - 10$  — the same answer.

**Practice (work this out).** Use the product rule to find the derivative of  $j(x) = 2x^5(4x^2 + x)$ .

## The Quotient Rule

Just as with products, the derivative of a quotient is *not* the quotient of the derivatives. The correct rule has a specific shape — and the order in the numerator matters, because of the minus sign.

**Definition 5** (The Quotient Rule). If  $f$  and  $g$  are differentiable and  $g(x) \neq 0$ , then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

In words: derivative of the top times the bottom, minus derivative of the bottom times the top, all over the bottom squared.

**Example 7** (Applying the quotient rule). Find the derivative of  $k(x) = \frac{5x^2}{4x+3}$ .

*Solution.* Let  $f(x) = 5x^2$  and  $g(x) = 4x + 3$ , so  $f'(x) = 10x$  and  $g'(x) = 4$ . Substitute into the rule:

$$k'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x+3) - 4(5x^2)}{(4x+3)^2}.$$

Now simplify the numerator:

$$k'(x) = \frac{40x^2 + 30x - 20x^2}{(4x+3)^2} = \frac{20x^2 + 30x}{(4x+3)^2}.$$

**Example 8** (A velocity from the quotient rule). The position of an object at time  $t$  is  $s(t) = \frac{t}{t^2+1}$ . Find the velocity at  $t = 0$ .

*Solution.* Velocity is  $s'(t)$ . With top  $t$  (derivative 1) and bottom  $t^2 + 1$  (derivative  $2t$ ):

$$s'(t) = \frac{1 \cdot (t^2 + 1) - 2t \cdot t}{(t^2 + 1)^2} = \frac{1 - t^2}{(t^2 + 1)^2}.$$

At  $t = 0$ :

$$s'(0) = \frac{1 - 0}{(0 + 1)^2} = 1.$$

## Bonus: Negative Powers and Combining Rules

The quotient rule quietly extends the power rule to *negative* integer exponents: the formula

$$\frac{d}{dx} x^k = k x^{k-1}$$

holds even when  $k$  is a negative integer. This often beats the quotient rule outright. For instance,

$$\frac{d}{dx} \left( \frac{6}{x^2} \right) = \frac{d}{dx} (6x^{-2}) = 6(-2)x^{-3} = -12x^{-3},$$

where we rewrote  $\frac{6}{x^2}$  as  $6x^{-2}$  first, then used the constant multiple and power rules — no quotient rule needed. In practice you seldom apply just one rule; you combine the sum, constant multiple, product, and quotient rules as the function demands, working from the outside in.

**Practice (work this out).** Find the derivative of  $g(x) = \frac{1}{x^7}$  by rewriting it with a negative exponent.

## Summary

- **Constant rule:**  $\frac{d}{dx}(c) = 0$ .
- **Power rule:**  $\frac{d}{dx} x^n = n x^{n-1}$  — drop the power in front, subtract one. (Also valid for negative integer powers.)
- **Constant multiple / sum / difference:** constants ride along, and you may differentiate term by term.

- **Product rule:**  $(fg)' = f'g + g'f$  — *not* the product of the derivatives.
- **Quotient rule:**  $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$  — mind the order and the minus sign.
- Real problems combine several rules; a tangent line needs a **point**  $f(a)$  and a **slope**  $f'(a)$ .
- Next time: the chain rule, for derivatives of composed functions.