

Derivatives as Rates of Change

Section 3.4

Calculus I — Lecture Notes

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The derivative is not just a slope on a graph — it is the speed of anything that changes.

We have spent weeks learning to compute derivatives. Today we cash them in. The derivative $f'(a)$ was defined as the *instantaneous rate of change* of f at $x = a$, and that phrase is the whole point: whenever a quantity changes — a position, a population, a company's cost — its derivative tells us how fast. In this lecture the same object f' plays four roles: it is a velocity, an acceleration, a growth rate, and a marginal cost. Learn to read the derivative in each language and you can answer questions algebra cannot touch.

The Amount of Change Formula

Suppose we know a quantity right now and we know how fast it is changing. Can we predict its value a moment later? Yes — and the tool is the derivative itself.

Definition 1. If f is defined on an interval $[a, a+h]$, the **amount of change** of f over the interval is $f(a+h) - f(a)$, and the **average rate of change** over that interval is the ratio

$$\frac{f(a+h) - f(a)}{h}.$$

We already know that the *instantaneous* rate of change of f at a is the derivative,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

For small enough h , the average rate is close to the instantaneous rate, $f'(a) \approx \frac{f(a+h) - f(a)}{h}$. Solving that for $f(a+h)$ gives the **amount of change formula**:

$$f(a+h) \approx f(a) + f'(a)h.$$

In words: the new value equals the old value plus the rate of change times the step. Geometrically we are riding the tangent line at $x = a$ a little distance h past the point, using it to estimate the true value.

Example 1 (Estimating the value of a function). If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.

Solution. We are at $a = 3$ and stepping to 3.2, so the step is $h = 3.2 - 3 = 0.2$. Apply the amount of change formula:

$$f(3.2) = f(3 + 0.2) \approx f(3) + f'(3)(0.2) = 2 + 5(0.2) = 2 + 1 = 3.$$

So $f(3.2) \approx 3$. We never needed a formula for f — one value and one rate were enough.

Practice (work this out). Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Motion Along a Line

The oldest use of the derivative is motion. Let $s(t)$ be the **position** of an object on a line at time t . The velocity is how fast position changes, and the acceleration is how fast velocity changes — so each is the derivative of the one before it.

Definition 2. Let $s(t)$ give the position of an object at time t . Then:

- the **velocity** is $v(t) = s'(t)$;
- the **speed** is $|v(t)|$, the magnitude of velocity;
- the **acceleration** is $a(t) = v'(t) = s''(t)$.

The sign of the velocity carries the direction: $v(t) > 0$ means the object moves in the positive direction, $v(t) < 0$ means the negative direction, and $v(t) = 0$ means the object is momentarily at rest. Speed throws the sign away — a car going -30 ft/s and one going $+30$ ft/s have the same speed, 30 ft/s.

Example 2 (Instantaneous versus average velocity). A ball is dropped from a height of 64 feet. Its height above the ground t seconds later is $s(t) = -16t^2 + 64$ (in feet).

- What is the instantaneous velocity of the ball when it hits the ground?
- What is its average velocity during the fall?

Solution. First find when the ball lands by setting $s(t) = 0$:

$$-16t^2 + 64 = 0 \implies t^2 = 4 \implies t = 2 \text{ s.}$$

- The velocity is $v(t) = s'(t) = -32t$. At the moment of impact,

$$v(2) = -32(2) = -64 \text{ ft/s.}$$

The sign is negative because the ball is moving downward; its speed is 64 ft/s.

- The average velocity over the fall is the amount of change in position divided by the elapsed time:

$$v_{\text{ave}} = \frac{s(2) - s(0)}{2 - 0} = \frac{0 - 64}{2} = -32 \text{ ft/s.}$$

The two numbers differ: the ball is slower than average early and faster than average late, which is exactly why we need the derivative for the instantaneous value.

Example 3 (Speeding up or slowing down). A particle moves along a coordinate axis with position $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$, and use them to decide (a) whether the particle is moving left or right at $t = 1$, and (b) whether it is speeding up or slowing down.

Solution. Differentiate twice:

$$v(t) = s'(t) = 3t^2 - 4, \quad a(t) = v'(t) = 6t.$$

Evaluate at $t = 1$:

$$v(1) = 3(1)^2 - 4 = -1, \quad a(1) = 6(1) = 6.$$

- Since $v(1) = -1 < 0$, the particle is moving in the negative direction — from right to left.
- Here $v(1) < 0$ and $a(1) > 0$, so velocity and acceleration point in *opposite* directions. The push is against the motion, so the speed $|v(t)|$ is decreasing: the particle is **slowing down**. (Rule of thumb: same signs on v and a mean speeding up; opposite signs mean slowing down.)

Example 4 (Finding when the particle is at rest and its direction). The position of a particle is $s(t) = t^3 - 9t^2 + 24t + 4$ for $t \geq 0$. Find the velocity, the time(s) when the particle is at rest, and the intervals on which it moves right and left.

Solution. The velocity is

$$v(t) = s'(t) = 3t^2 - 18t + 24 = 3(t^2 - 6t + 8) = 3(t - 2)(t - 4).$$

The particle is at rest when $v(t) = 0$, i.e. at $t = 2$ and $t = 4$. These two times split $[0, \infty)$ into three pieces; test the sign of v on each:

$$v(1) = 3(-1)(-3) = 9 > 0, \quad v(3) = 3(1)(-1) = -3 < 0, \quad v(5) = 3(3)(1) = 27 > 0.$$

So $v(t) > 0$ on $[0, 2)$ and $(4, \infty)$ — the particle moves from left to right there — and $v(t) < 0$ on $(2, 4)$, where it moves from right to left. It reverses direction at $t = 2$ and $t = 4$.

Practice (work this out). A particle moves along a coordinate axis with position $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at $t = 3$?

Population Change

Velocity was the rate of change of position; a growth rate is the rate of change of a population. Same derivative, new costume.

Definition 3. If $P(t)$ is the number of individuals in a population at time t , then the **population growth rate** is $P'(t)$.

Because P' is a rate of change, the amount of change formula applies verbatim: knowing today's population and today's growth rate lets us estimate the population in the near future.

Example 5 (Estimating a population). The population of a city is tripling every 5 years. If its current population is 10,000, what will the population be approximately 2 years from now?

Solution. Let $P(t)$ be the population in thousands, t years from now, so $P(0) = 10$. Tripling in 5 years means $P(5) = 30$. We do not have a formula for P , so estimate the current growth rate by the average rate of change:

$$P'(0) \approx \frac{P(5) - P(0)}{5 - 0} = \frac{30 - 10}{5} = 4 \text{ (thousand per year)}.$$

Now apply the amount of change formula with $a = 0$ and $h = 2$:

$$P(2) \approx P(0) + P'(0)(2) = 10 + 4(2) = 18.$$

In thousands, that is a population of about 18,000 in two years.

Changes in Cost and Revenue

Economists give the derivative yet another name. If it costs $C(x)$ to produce x items, the derivative $C'(x)$ is called the **marginal cost**. Likewise the derivative of revenue is marginal revenue, and the derivative of profit is marginal profit.

Definition 4. Let x be the number of items produced or sold.

- If $C(x)$ is the cost, the **marginal cost** is $MC(x) = C'(x)$.
- If $R(x)$ is the revenue, the **marginal revenue** is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit, the **marginal profit** is $MP(x) = P'(x) = R'(x) - C'(x)$.

Why is this useful? Take $h = 1$ (one extra item) in the average rate of change:

$$C'(x) \approx \frac{C(x+1) - C(x)}{1} = C(x+1) - C(x).$$

So the marginal cost $C'(x)$ is approximately the cost of producing one *more* item beyond the x you already made. The same reading holds for revenue and profit — the marginal function estimates the effect of the next single unit.

Example 6 (Applying marginal revenue). The number of barbeque dinners sold, x , relates to the price charged, p , by $p(x) = 9 - 0.03x$ for $0 \leq x \leq 300$. So the revenue from selling x dinners is

$$R(x) = x p(x) = x(9 - 0.03x) = -0.03x^2 + 9x.$$

Use marginal revenue to estimate the revenue from selling the 101st dinner, and compare with the actual revenue.

Solution. The marginal revenue is

$$MR(x) = R'(x) = -0.06x + 9.$$

The 101st dinner is the next one after $x = 100$, so evaluate there:

$$R'(100) = -0.06(100) + 9 = -6 + 9 = 3.$$

The marginal revenue estimates about \$3 from the 101st dinner. The actual revenue is the difference

$$R(101) - R(100) = (-0.03(101)^2 + 9(101)) - (-0.03(100)^2 + 9(100)) = 602.97 - 600 = 2.97,$$

that is, \$2.97. The estimate of \$3 is close, and far easier to compute.

Practice (work this out). A publisher's cost, in dollars, of producing x thousand copies of a book is $C(x) = -0.02x^2 + 8x + 40$. Find the marginal cost function $C'(x)$ and use it to estimate the cost of producing the 6th thousand copies (i.e. near $x = 5$).

Summary

- The derivative $f'(a)$ is the instantaneous rate of change of f at a . Every application today is one reading of this single idea.
- **Amount of change formula:** $f(a+h) \approx f(a) + f'(a)h$ — old value plus rate times step; it rides the tangent line to predict a nearby value.
- **Motion:** position $s(t)$, velocity $v(t) = s'(t)$, speed $|v(t)|$, acceleration $a(t) = s''(t)$. Sign of v gives direction; v and a with the same sign means speeding up, opposite signs means slowing down.
- **Populations:** the growth rate is $P'(t)$; combine a current count with a current rate to estimate the future.
- **Economics:** marginal cost/revenue/profit are C' , R' , P' , each approximating the effect of one more unit.
- Next time: related rates — linking the rates of change of two quantities that move together.