

Derivatives of Trigonometric Functions

Section 3.5

Calculus I — Lecture Notes

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Anything that oscillates — a spring, a heartbeat, a sound wave — is trig in disguise, and its motion is a derivative.

Every rule so far has been about powers of x : the power rule, the product rule, the quotient rule. But much of the physical world does not go in a straight line — it *repeats*. A mass on a spring bobs up and down, a pendulum swings back and forth, a wave rolls in. These are described by $\sin$ and $\cos$, and to find velocity and acceleration we need their derivatives. Today we compute the derivative of all six trig functions, and we do it starting from a single beautiful limit.

Two Key Trigonometric Limits

Everything today rests on two limits. They cannot be found by plugging in $h = 0$ (both give $\frac{0}{0}$), but they have clean values.

Definition 1 (The two fundamental trig limits).

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0.$$

Here h is measured in **radians**. (This is the entire reason calculus is done in radians, not degrees.)

The first limit says that for a small angle, $\sin h$ is almost exactly h — the arc and its vertical drop are nearly equal. The second says $\cos h$ flattens out to 1 so fast that its gap from 1 is small even compared to h . Keep both in your pocket; the derivative formulas fall right out of them.

Derivatives of Sine and Cosine

Start from the limit definition applied to $f(x) = \sin x$, and use the angle-addition identity $\sin(x + h) = \sin x \cos h + \cos x \sin h$.

Definition 2 (Derivatives of $\sin x$ and $\cos x$).

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x.$$

The derivative of sine is cosine; the derivative of cosine is negative sine (mind the sign).

Example 1 (Deriving $\frac{d}{dx} \sin x$ from the definition). Show that $\frac{d}{dx}(\sin x) = \cos x$.

Solution. Apply the limit definition and the identity for $\sin(x + h)$:

$$\begin{aligned}\frac{d}{dx} \sin x &= \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) = \lim_{h \rightarrow 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) \\ &= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.\end{aligned}$$

The two fundamental limits do all the work in the last step. The same technique gives $\frac{d}{dx}(\cos x) = -\sin x$.

A picture confirms it: where $\sin x$ has a horizontal tangent (a peak or a valley), $\cos x$ is zero; where $\sin x$ is rising, $\cos x > 0$; where $\sin x$ is falling, $\cos x < 0$. The derivative graph is exactly the cosine curve.

Example 2 (A product with $\sin x$). Find the derivative of $f(x) = 5x^3 \sin x$.

Solution. This is a product, so use the product rule with the new fact $\frac{d}{dx}(\sin x) = \cos x$:

$$f'(x) = \frac{d}{dx}(5x^3) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 = 15x^2 \sin x + 5x^3 \cos x.$$

Example 3 (A quotient with $\cos x$). Find the derivative of $g(x) = \frac{\cos x}{4x^2}$.

Solution. Apply the quotient rule, remembering $\frac{d}{dx}(\cos x) = -\sin x$:

$$g'(x) = \frac{(-\sin x)(4x^2) - (\cos x)(8x)}{(4x^2)^2} = \frac{-4x^2 \sin x - 8x \cos x}{16x^4}.$$

Cancel a factor of $4x$ from top and bottom:

$$g'(x) = \frac{-x \sin x - 2 \cos x}{4x^3}.$$

Practice (work this out). Find the derivative of $f(x) = \sin x \cos x$.

Derivatives of the Other Four Trig Functions

The remaining four functions are just quotients of sine and cosine, so the quotient rule finishes the job.

Example 4 (The derivative of $\tan x$). Find the derivative of $f(x) = \tan x$.

Solution. Write $\tan x = \frac{\sin x}{\cos x}$ and apply the quotient rule:

$$f'(x) = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

By the Pythagorean identity $\cos^2 x + \sin^2 x = 1$, this is $\frac{1}{\cos^2 x}$, and since $\sec x = \frac{1}{\cos x}$,

$$f'(x) = \sec^2 x.$$

Running the same quotient-rule computation on $\cot x$, $\sec x$, and $\csc x$ gives the full table.

Definition 3 (Derivatives of $\tan x$, $\cot x$, $\sec x$, $\csc x$).

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \sec^2 x, & \frac{d}{dx}(\cot x) &= -\csc^2 x, \\ \frac{d}{dx}(\sec x) &= \sec x \tan x, & \frac{d}{dx}(\csc x) &= -\csc x \cot x.\end{aligned}$$

The three “co-” functions ($\cos$, $\cot$, $\csc$) all pick up a minus sign.

Example 5 (Equation of a tangent line). Find the equation of the line tangent to $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution. We need a point and a slope. The point:

$$f\left(\frac{\pi}{4}\right) = \cot \frac{\pi}{4} = 1,$$

so the line passes through $(\frac{\pi}{4}, 1)$. The slope is f' evaluated there, using $\frac{d}{dx}(\cot x) = -\csc^2 x$:

$$f'(x) = -\csc^2 x, \quad f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -(\sqrt{2})^2 = -2.$$

Point-slope form gives $y - 1 = -2(x - \frac{\pi}{4})$, or equivalently

$$y = -2x + 1 + \frac{\pi}{2}.$$

Practice (work this out). Find the derivative of $f(x) = \csc x + x \tan x$.

Higher-Order Derivatives and a Repeating Pattern

Because differentiating cycles $\sin \rightarrow \cos \rightarrow -\sin \rightarrow -\cos \rightarrow \sin$, the derivatives of $\sin x$ repeat with period 4.

$$y = \sin x, \quad \frac{dy}{dx} = \cos x, \quad \frac{d^2y}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \quad \frac{d^4y}{dx^4} = \sin x.$$

After four steps we are back where we started. To find a very high derivative, just divide the order by 4 and look at the remainder.

Example 6 (Using the pattern). Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution. Divide 74 by 4: $74 = 4(18) + 2$, remainder 2. So the 74th derivative matches the 2nd derivative in the cycle:

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

Application: Simple Harmonic Motion

If a particle's position is $s(t)$, then velocity is $v(t) = s'(t)$ and acceleration is $a(t) = v'(t) = s''(t)$. When $s(t)$ is built from $\sin$ and $\cos$, the motion oscillates — this is **simple harmonic motion**, the physics of springs and pendulums.

Example 7 (When is the particle at rest?). A particle moves along a line with position $s(t) = 2 \sin t - t$ for $0 \leq t \leq 2\pi$. At what times is it at rest?

Solution. The particle is at rest when its velocity is zero. Differentiate:

$$v(t) = s'(t) = 2 \cos t - 1.$$

Set $v(t) = 0$: $2 \cos t - 1 = 0 \Rightarrow \cos t = \frac{1}{2}$. On $0 \leq t \leq 2\pi$ the solutions are

$$t = \frac{\pi}{3} \quad \text{and} \quad t = \frac{5\pi}{3}.$$

The particle is at rest at those two times.

Example 8 (Speeding up or slowing down?). A particle has position $s(t) = 2 - \sin t$. Find $v(\frac{\pi}{4})$ and $a(\frac{\pi}{4})$, and decide whether it is speeding up or slowing down.

Solution. Differentiate once for velocity, again for acceleration:

$$v(t) = s'(t) = -\cos t, \quad a(t) = v'(t) = \sin t.$$

Evaluate at $t = \frac{\pi}{4}$:

$$v(\frac{\pi}{4}) = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}, \quad a(\frac{\pi}{4}) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Velocity is negative and acceleration is positive: they point in opposite directions, so the acceleration is fighting the motion. The particle is **slowing down**.

Practice (work this out). A block on a spring has position $s(t) = 2 \sin t$. Find $v(\frac{5\pi}{6})$ and $a(\frac{5\pi}{6})$, and decide whether the block is speeding up or slowing down.

Summary

- Two limits power everything: $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$ and $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$ (angles in radians).
- The core pair: $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$.
- The other four, from the quotient rule: $\tan x \rightarrow \sec^2 x$, $\cot x \rightarrow -\csc^2 x$, $\sec x \rightarrow \sec x \tan x$, $\csc x \rightarrow -\csc x \cot x$. Every “co-” function carries a minus sign.
- Combine with the product and quotient rules to differentiate expressions like $5x^3 \sin x$ or $\frac{\cos x}{4x^2}$.
- Derivatives of $\sin x$ repeat every 4 steps; use the remainder mod 4 for high-order derivatives.
- Application: for $s(t)$ built from $\sin$ and $\cos$, $v = s'$ and $a = s''$ describe simple harmonic motion — at rest when $v = 0$, slowing down when v and a have opposite signs.