

The Chain Rule and Implicit Differentiation

Sections 3.6 & 3.8

Calculus I — Lecture Notes

September 29, 2026

The Missing Rule

We can differentiate the pieces — now we learn to differentiate the machine built from them.

We already know how to differentiate the basic functions (x^n , $\sin x$, $\cos x$, and so on), and we can handle sums, differences, products, quotients, and constant multiples. But none of those tools touches a function like

$$h(x) = \sin(x^3) \quad \text{or} \quad k(x) = (3x^2 + 1)^5.$$

These are **compositions**: an inner function feeds into an outer function. The rule that differentiates a composition is the **chain rule**, and it is the single most-used differentiation rule in the whole course. It says, roughly: *the derivative of the outside times the derivative of the inside*.

Stating the Chain Rule

Recall composition: $(f \circ g)(x) = f(g(x))$ means “do g first, then feed the result into f .” Here g is the **inner** function and f is the **outer** function. The chain rule tells us how the rates of change multiply.

Definition 1 (The Chain Rule). Let f and g be functions, with g differentiable at x and f differentiable at $g(x)$. Then the composite $h(x) = (f \circ g)(x) = f(g(x))$ is differentiable at x , and

$$h'(x) = f'(g(x)) g'(x).$$

In words: the derivative of the *outer* function evaluated at the inner function, times the derivative of the *inner* function.

There is a second, equally useful way to write this. If y is a function of u , and u is a function of x , then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

This is the **Leibniz form**. It is easy to remember because it looks like the du ’s “cancel” — a helpful piece of bookkeeping, even though these are not really fractions. The prime form and the Leibniz form say exactly the same thing.

Strategy. To differentiate $h(x) = f(g(x))$: identify the inner $g(x)$ and outer $f(x)$; compute $f'(g(x))$ (derivative of the outside, inside left alone); multiply by $g'(x)$ (derivative of the inside).

Work from the outside in, and never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

The most common composition is a function raised to a power: $h(x) = (g(x))^n$. Taking $f(x) = x^n$ as the outer function, so that $f'(x) = nx^{n-1}$, the chain rule gives a rule worth memorizing on its own.

Definition 2 (General Power Rule). For all x where the derivative is defined, if $h(x) = (g(x))^n$, then

$$h'(x) = n (g(x))^{n-1} g'(x).$$

That is: bring down the power, drop the exponent by one *leaving the inside untouched*, then multiply by the derivative of the inside.

Example 1 (Chain and power rules). Find the derivative of $h(x) = \frac{1}{(3x^2 + 1)^2}$.

Solution. First rewrite the quotient as a power so the power rule applies:

$$h(x) = (3x^2 + 1)^{-2}.$$

The inner function is $g(x) = 3x^2 + 1$, with $g'(x) = 6x$. Apply the general power rule with $n = -2$:

$$h'(x) = -2(3x^2 + 1)^{-3} (6x) = -12x(3x^2 + 1)^{-3}.$$

Rewriting back in the original form,

$$h'(x) = \frac{-12x}{(3x^2 + 1)^3}.$$

Example 2 (Power rule with a trig inside). Find the derivative of $h(x) = \sin^3 x$.

Solution. The notation $\sin^3 x$ means $(\sin x)^3$, so this is a power of an inner function. Here the inner function is $g(x) = \sin x$, with $g'(x) = \cos x$. Apply the general power rule with $n = 3$:

$$h'(x) = 3(\sin x)^2 \cos x = 3 \sin^2 x \cos x.$$

Notice the inside $(\sin x)$ is left completely alone in the first factor; only then do we multiply by its derivative $\cos x$.

Combining the Chain Rule with Trig

The chain rule combines with the derivatives of the trigonometric functions. For a general inner function $g(x)$,

$$\frac{d}{dx} [\sin(g(x))] = \cos(g(x)) g'(x), \quad \frac{d}{dx} [\cos(g(x))] = -\sin(g(x)) g'(x),$$

$$\frac{d}{dx} [\tan(g(x))] = \sec^2(g(x)) g'(x), \quad \frac{d}{dx} [\sec(g(x))] = \sec(g(x)) \tan(g(x)) g'(x).$$

You do not need to memorize these as separate facts — each is just the chain rule applied to a trig derivative you already know.

Example 3 (A trig function of a polynomial). Find the derivative of $h(x) = \cos(5x^2)$.

Solution. The outer function is $\cos$, the inner is $g(x) = 5x^2$ with $g'(x) = 10x$. Since the derivative of $\cos u$ is $-\sin u$,

$$h'(x) = -\sin(5x^2) \cdot 10x = -10x \sin(5x^2).$$

Example 4 (Chain rule with the product rule). Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

Solution. This is a *product* of two composite functions, so we start with the product rule and apply the chain rule to each factor.

$$\begin{aligned} h'(x) &= \frac{d}{dx}[(2x+1)^5] \cdot (3x-2)^7 + \frac{d}{dx}[(3x-2)^7] \cdot (2x+1)^5 && \text{Product rule.} \\ &= 5(2x+1)^4 \cdot 2 \cdot (3x-2)^7 + 7(3x-2)^6 \cdot 3 \cdot (2x+1)^5 && \text{Chain rule on each.} \\ &= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.} \\ &= (2x+1)^4(3x-2)^6[10(3x-2) + 21(2x+1)] && \text{Factor out common terms.} \\ &= (2x+1)^4(3x-2)^6(72x+1). && \text{Simplify.} \end{aligned}$$

Composites of Three or More Functions

When a function is a composition of *three* functions, we apply the chain rule twice — once for each layer, working from the outside in. If $k(x) = h(f(g(x)))$, then

$$k'(x) = h'(f(g(x))) f'(g(x)) g'(x).$$

The derivative of a composition of three functions has three parts; four functions, four parts; and so on. You never need to memorize this — just peel one layer at a time.

Example 5 (A three-layer composition). Find the derivative of $k(x) = \cos^4(7x^2 + 1)$.

Solution. Rewrite so the layers are visible:

$$k(x) = \left(\cos(7x^2 + 1) \right)^4.$$

The three layers, outside to inside, are: raise to the 4th power; take cosine; and $7x^2 + 1$. Apply the chain rule one layer at a time.

$$\begin{aligned} k'(x) &= 4 \left(\cos(7x^2 + 1) \right)^3 \cdot \frac{d}{dx} \left[\cos(7x^2 + 1) \right] && \text{Power rule (outer layer).} \\ &= 4 \left(\cos(7x^2 + 1) \right)^3 (-\sin(7x^2 + 1)) \cdot \frac{d}{dx} [7x^2 + 1] && \text{Chain rule (cosine layer).} \\ &= 4 \left(\cos(7x^2 + 1) \right)^3 (-\sin(7x^2 + 1))(14x) && \text{Differentiate the inside.} \\ &= -56x \sin(7x^2 + 1) \cos^3(7x^2 + 1). && \text{Simplify.} \end{aligned}$$

The Leibniz Form in Practice

The Leibniz form $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ is handy when you want to name the inner quantity explicitly. Set u equal to the inside, compute the two easy derivatives, multiply, then rewrite the answer entirely in terms of the original variable.

Example 6 (Leibniz notation). Find the derivative of $y = \tan(4x^2 - 3x + 1)$.

Solution. Let $u = 4x^2 - 3x + 1$, so that $y = \tan u$. The two pieces are

$$\frac{du}{dx} = 8x - 3 \quad \text{and} \quad \frac{dy}{du} = \sec^2 u.$$

Multiply them, then substitute u back:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot (8x - 3) = \sec^2(4x^2 - 3x + 1) (8x - 3).$$

The final answer is written entirely in x , as it must be.

Practice Problems

Practice (work this out). Find the derivative of $h(x) = (2x^3 + 2x - 1)^4$.

Practice (work this out). Use the chain rule to find the derivative of $h(x) = \sin(7x + 2)$, and separately find the derivative of $g(x) = \cos^3(\pi x)$.

Implicit Differentiation

Some curves refuse to be a single function — we can still find their slopes.

Up to now every curve we differentiated came pre-packaged as $y = f(x)$: the output y was written *explicitly* in terms of x , and we differentiated the formula. But many important curves are not functions at all. The circle $x^2 + y^2 = 25$ fails the vertical line test — most vertical lines hit it *twice*. There is no single $y = f(x)$ describing the whole thing. Yet the circle plainly has a tangent line at $(3, -4)$ with a definite slope. We now learn a technique, **implicit differentiation**, that finds $\frac{dy}{dx}$ directly from an equation relating x and y , without ever solving for y .

Explicit vs. Implicit

Definition 3. When y is written by itself on one side, as $y = f(x)$, we say y is an **explicit** function of x . When x and y are tangled together in an equation that is not solved for y , we say the equation defines y **implicitly** in terms of x .

For example, $y = x^2 + 1$ is explicit. The same relationship written as $y - x^2 = 1$ defines y implicitly. In that easy case we could solve for y and be done. The point of today is the cases where we *cannot* conveniently solve — or where the equation packs several functions into one picture.

A single equation can define *many* functions implicitly. The circle $x^2 + y^2 = 25$ contains both the top half $y = \sqrt{25 - x^2}$ and the bottom half $y = -\sqrt{25 - x^2}$, and any number of pieced-together curves besides. Rather than pick one branch and solve for it, we treat y as *some* differentiable function of x sitting inside the equation, and differentiate.

The Key Idea: y Is a Function of x

Everything rests on one observation. Since y stands for a function of x , any expression built from y is a *composition*, so differentiating it requires the **chain rule**. Differentiating with respect to x :

$$\frac{d}{dx}(x^2) = 2x, \quad \text{but} \quad \frac{d}{dx}(y^2) = 2y \frac{dy}{dx}.$$

The extra factor $\frac{dy}{dx}$ is the chain rule at work — the derivative of the inside function y . Similarly $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$ and $\frac{d}{dx}(y^3) = 3y^2 \frac{dy}{dx}$. Forgetting that factor is the one mistake to guard against all lecture.

Definition 4 (Implicit Differentiation — the four steps). To find $\frac{dy}{dx}$ from an equation relating x and y :

1. Differentiate *both* sides with respect to x , remembering that y is a function of x (so every y -term gets a factor of $\frac{dy}{dx}$).
2. Collect all terms containing $\frac{dy}{dx}$ on the left; move everything else to the right.
3. Factor out $\frac{dy}{dx}$ on the left.
4. Divide to solve for $\frac{dy}{dx}$.

Worked Examples

Example 7 (The circle). Assuming y is defined implicitly by $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution. Differentiate both sides with respect to x . The left side is a sum; differentiate term by term, using the chain rule on y^2 :

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25), \quad \implies \quad 2x + 2y \frac{dy}{dx} = 0.$$

The right side is the derivative of a constant, which is 0. Now isolate $\frac{dy}{dx}$: move $2x$ to the right and divide by $2y$,

$$2y \frac{dy}{dx} = -2x \quad \implies \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Notice the answer involves *both* x and y . That is normal for implicit differentiation — and it is exactly what we want, since a given x on the circle can have two different y 's with opposite slopes.

Example 8 (Product rule and a trig term). Assuming y is defined implicitly by $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$.

Solution. Differentiate both sides. The term $x^3 \sin y$ is a product of x^3 and $\sin y$, so it needs the **product rule**; and $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$ by the chain rule:

$$\underbrace{\left(3x^2 \cdot \sin y + x^3 \cdot \cos y \frac{dy}{dx} \right)}_{\text{product rule on } x^3 \sin y} + \frac{dy}{dx} = 4.$$

Gather the $\frac{dy}{dx}$ terms on the left and push $3x^2 \sin y$ to the right:

$$x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y.$$

Factor out $\frac{dy}{dx}$, then divide:

$$\frac{dy}{dx}(x^3 \cos y + 1) = 4 - 3x^2 \sin y \quad \implies \quad \frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}.$$

Example 9 (Tangent line to a circle). Find an equation of the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution. We already found $\frac{dy}{dx} = -\frac{x}{y}$ for this circle. The slope of the tangent at a point is that expression *evaluated at the point*. At $(3, -4)$,

$$\left. \frac{dy}{dx} \right|_{(3, -4)} = -\frac{3}{-4} = \frac{3}{4}.$$

Use point-slope form with the point $(3, -4)$ and slope $\frac{3}{4}$:

$$y - (-4) = \frac{3}{4}(x - 3) \quad \implies \quad y = \frac{3}{4}x - \frac{25}{4}.$$

This one line touches the circle at $(3, -4)$ — something no amount of algebra on the equation alone would hand us so cleanly.

Example 10 (The folium of Descartes). Find an equation of the line tangent to the curve $y^3 + x^3 - 3xy = 0$ at the point $(\frac{3}{2}, \frac{3}{2})$.

Solution. Differentiate every term with respect to x . The term $3xy$ is a product of x and y , so it needs the product rule; y^3 needs the chain rule:

$$3y^2 \frac{dy}{dx} + 3x^2 - 3\left(y + x \frac{dy}{dx}\right) = 0.$$

Distribute the -3 and separate the $\frac{dy}{dx}$ terms:

$$3y^2 \frac{dy}{dx} - 3x \frac{dy}{dx} = 3y - 3x^2 \quad \implies \quad \frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2.$$

Divide to solve, and cancel the common 3:

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}.$$

Now evaluate at $(\frac{3}{2}, \frac{3}{2})$:

$$\left. \frac{dy}{dx} \right|_{(3/2, 3/2)} = \frac{\frac{3}{2} - \frac{9}{4}}{\frac{9}{4} - \frac{3}{2}} = \frac{-\frac{3}{4}}{\frac{3}{4}} = -1.$$

With slope -1 through $(\frac{3}{2}, \frac{3}{2})$, point-slope gives

$$y - \frac{3}{2} = -1\left(x - \frac{3}{2}\right) \quad \implies \quad y = -x + 3.$$

A Second Derivative

The technique iterates: once we have $\frac{dy}{dx}$, we can differentiate again to get $\frac{d^2y}{dx^2}$, remembering to substitute back for $\frac{dy}{dx}$ when it reappears.

Example 11 (Second derivative on the circle). Find $\frac{d^2y}{dx^2}$ for $x^2 + y^2 = 25$.

Solution. We found $\frac{dy}{dx} = -\frac{x}{y}$. Differentiate that with respect to x using the **quotient rule** (treating y as a function of x):

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right) = -\frac{(1) \cdot y - x \cdot \frac{dy}{dx}}{y^2} = \frac{-y + x \frac{dy}{dx}}{y^2}.$$

Substitute $\frac{dy}{dx} = -\frac{x}{y}$:

$$\frac{d^2y}{dx^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y - \frac{x^2}{y}}{y^2} = \frac{-y^2 - x^2}{y^3}.$$

Since the point lies on the circle, $x^2 + y^2 = 25$, so the numerator is -25 and

$$\frac{d^2y}{dx^2} = -\frac{25}{y^3}.$$

Practice

Practice (work this out). Assuming y is defined implicitly by $6x^2 + 3y^2 = 12$, find $\frac{dy}{dx}$.

Practice (work this out). Find an equation of the line tangent to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

Summary

Chain rule.

- The **chain rule** differentiates a composition $h(x) = f(g(x))$: derivative of the outside evaluated at the inside, times the derivative of the inside, $h'(x) = f'(g(x))g'(x)$.
- **Leibniz form**: if y depends on u and u on x , then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ — same rule, different bookkeeping.
- **General power rule**: $\frac{d}{dx}(g(x))^n = n(g(x))^{n-1}g'(x)$ — leave the inside alone, then multiply by its derivative.
- Combines cleanly with trig, the product rule, and the quotient rule; apply one rule at a time.
- For **three or more** layers, apply the chain rule repeatedly, working from the outside in — one derivative at a time.
- Golden rule: *outside first, then multiply by the derivative of the inside*, and never evaluate a derivative at a derivative.

Implicit differentiation.

- An equation in x and y may define y **implicitly** even when it is not a function and cannot be solved for y (e.g. the circle $x^2 + y^2 = 25$).
- **Implicit differentiation:** differentiate both sides with respect to x , treating y as a function of x — so every y -term picks up a factor $\frac{dy}{dx}$ from the chain rule.
- Then collect the $\frac{dy}{dx}$ terms, factor, and divide. The answer typically involves *both* x and y .
- Watch for the **product rule** (terms like xy or $x^3 \sin y$) and the **chain rule** (terms like y^2 , y^3 , $\sin y$).
- For a **tangent line**, evaluate $\frac{dy}{dx}$ at the given point to get the slope, then use point-slope form.
- The method iterates: differentiate again (with the quotient rule and a back-substitution) to reach the **second derivative**.