The Limit Laws

Section 2.3

Calculus I — Lecture Notes

September 3, 2026

From Guessing to Computing

Last time we guessed *limits from graphs and tables. Today we* compute *them.*

Building a table of values or squinting at a graph told us what a limit *looked like*, but it was slow and never quite certain — how do we know the next decimal place wouldn't surprise us? In this lecture we replace that guesswork with a small set of algebra rules, the **limit laws**. They let us break a complicated limit into simple pieces, evaluate each piece, and reassemble the answer exactly. For most functions this collapses to a single move: plug the number in. The rest of the lecture is about the cases where plugging in fails — the dreaded 0/0 — and the clever rewrites that rescue them.

The Basic Limits and the Limit Laws

Everything is built from two limits so simple they barely look like limits.

Definition 1 (Basic Limit Results)**.** For any real number a and any constant c,

$$\lim_{x\to a} x = a \qquad \text{and} \qquad \lim_{x\to a} c = c.$$

The first says that as x marches toward a, the output (which *is* x) marches toward a. The second says a constant function never moves, so its limit is that constant. From these two seeds and the laws below, we can grow the limit of almost any algebraic function.

Definition 2 (The Limit Laws)**.** Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist, and let c be a constant. Then:

- **Sum/Difference:** $\lim_{x\to a} \big(f(x) \pm g(x)\big) = L \pm M$.
- **Constant multiple:** $\lim_{x\to a} c\,f(x) = c\,L$.
- **Product:** $\lim_{x\to a} f(x)\,g(x) = L \cdot M$.
- **Quotient:** $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, provided $M \neq 0$.
- **Power:** $\lim_{x\to a} \big(f(x)\big)^n = L^n$ for every positive integer n.
- **Root:** $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$, valid for all L if n is odd, and for $L \geq 0$ if n is even.

In words: the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. The catch that makes each law honest: the pieces you split into must *themselves* have limits. Watch that the denominator does not limit to 0 before you use the quotient law.

Example 1 (Applying the laws one step at a time)**.** Use the limit laws to evaluate $\lim_{x\to -3}(4x+2)$.

Solution. We apply one law per line so it is clear which is doing the work.

$$\begin{aligned}
\lim_{x\to -3}(4x+2) &= \lim_{x\to -3} 4x + \lim_{x\to -3} 2 && \text{(sum law)}\\
&= 4\cdot \lim_{x\to -3} x + \lim_{x\to -3} 2 && \text{(constant multiple law)}\\
&= 4\cdot(-3)+2 = -10. && \text{(basic limits)}
\end{aligned}$$

Example 2 (Using the laws repeatedly)**.** Evaluate $\lim_{x\to 2}\dfrac{2x^2-3x+1}{x^3+4}$.

Solution. First check the denominator's limit is nonzero: at $x=2$ it heads to $2^3+4=12\neq 0$, so the quotient law is legal.

$$\begin{aligned}
\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4} &= \frac{\lim_{x\to 2}\left(2x^2-3x+1\right)}{\lim_{x\to 2}\left(x^3+4\right)} && \text{(quotient law)}\\
&= \frac{2\left(\lim_{x\to 2} x\right)^2 - 3\lim_{x\to 2} x + \lim_{x\to 2} 1}{\left(\lim_{x\to 2} x\right)^3 + \lim_{x\to 2} 4} && \text{(sum, constant multiple, power)}\\
&= \frac{2(2)^2-3(2)+1}{(2)^3+4} = \frac{3}{12} = \frac{1}{4}. && \text{(basic limits)}
\end{aligned}$$

Direct Substitution

Look closely at the last two examples: in each one, the final step was nothing more than plugging a into the formula. That is not a coincidence.

Definition 3 (Limits of Polynomials and Rational Functions)**.** Let $p(x)$ and $q(x)$ be polynomials and let a be any real number. Then

$$\lim_{x\to a} p(x) = p(a), \qquad \lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \quad \text{whenever } q(a)\neq 0.$$

This is the payoff of the limit laws: for a polynomial, or a rational function at a point where the denominator isn't zero, **the limit is just the value**. This is called evaluating by **direct substitution**. You should reach for it first, every time.

Example 3 (Direct substitution)**.** Evaluate $\lim_{x\to 3}\dfrac{2x^2-3x+1}{5x+4}$.

Solution. The denominator at $x=3$ is $5(3)+4=19\neq 0$, so this rational function is defined at 3 and we simply substitute:

$$\lim_{x\to 3}\frac{2x^2-3x+1}{5x+4} = \frac{2(3)^2-3(3)+1}{5(3)+4} = \frac{18-9+1}{19} = \frac{10}{19}.$$

Practice (work this out). Evaluate $\lim_{x\to -2}\left(3x^3-2x+7\right)$ by direct substitution.

When Substitution Fails: the Indeterminate Form 0/0

Direct substitution is the whole game — until it isn't. Substitute into $\lim_{x \to 1} \dfrac{x^2-1}{x-1}$ and you get $\frac{0}{0}$. This is called an **indeterminate form**: it is *not* an answer and it is *not* automatically "undefined." The 0 on top is trying to push the fraction to 0; the 0 on the bottom is trying to blow it up. Who wins is genuinely undecided until we do more work.

Here is the key idea that unlocks every such problem. If two functions agree at every x near a — everywhere *except possibly at a itself* — then they have the same limit at a. The limit never looks at the single point $x = a$, only at the neighbors. So we are free to cancel the offending factor and take the limit of the simpler function.

Strategy for the form 0/0: rewrite the fraction so the trouble cancels, then substitute.

- If top and bottom are polynomials, **factor and cancel** the common factor.
- If a square root is involved, multiply by the **conjugate** to clear it.
- Only after canceling do you apply direct substitution.

Example 4 (The factor-and-cancel technique)**.** Evaluate $\lim_{x \to 3} \dfrac{x^2-3x}{2x^2-5x-3}$.

Solution. **Step 1.** Substitute $x = 3$: the top is $9 - 9 = 0$ and the bottom is $18 - 15 - 3 = 0$, so we have the form 0/0. Factoring is the move.

$$\frac{x^2-3x}{2x^2-5x-3} = \frac{x(x-3)}{(x-3)(2x+1)}.$$

Step 2. For every $x \neq 3$ the common factor $(x-3)$ cancels, so the original function agrees with $\dfrac{x}{2x+1}$ near 3:

$$\lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)} = \lim_{x \to 3} \frac{x}{2x+1}.$$

Step 3. Now the denominator at 3 is $7 \neq 0$, so substitute:

$$\lim_{x \to 3} \frac{x}{2x+1} = \frac{3}{2(3)+1} = \frac{3}{7}.$$

Example 5 (The rationalize (conjugate) technique)**.** Evaluate $\lim_{x \to -1} \dfrac{\sqrt{x+2}-1}{x+1}$.

Solution. **Step 1.** At $x = -1$ the top is $\sqrt{1} - 1 = 0$ and the bottom is 0: the form is 0/0. A square-root difference calls for its **conjugate**, $\sqrt{x+2}+1$. Multiply top and bottom by it:

$$\frac{\sqrt{x+2}-1}{x+1} \cdot \frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}.$$

Step 2. Multiply out the *numerator* only — we want to keep $(x+1)$ factored in the bottom, hoping it cancels. Since $(\sqrt{x+2}-1)(\sqrt{x+2}+1) = (x+2) - 1 = x+1$,

$$= \frac{x+1}{(x+1)\big(\sqrt{x+2}+1\big)}.$$

Step 3. Cancel the common $(x+1)$:

$$= \frac{1}{\sqrt{x+2}+1}.$$

Step 4. Now substitute $x = -1$:

$$\lim_{x \to -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

Practice (work this out). Evaluate $\displaystyle\lim_{x \to -3} \frac{x^2+4x+3}{x^2-9}$ by factoring and canceling.

The Squeeze Theorem

The techniques above handle algebraic functions beautifully, but they stall on functions like $\cos x$ that we cannot factor. For these we need one more idea. Suppose a function we care about is trapped between two functions we understand, and those two outer functions are being funneled toward the *same* value. Then the trapped function has no room to escape — it must go to that value too.

Definition 4 (The Squeeze Theorem)**.** Let f, g, and h be defined for all $x \neq a$ on an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

near a, and if

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x),$$

then $\displaystyle\lim_{x \to a} g(x) = L$ as well.

The name is the picture: g is squeezed between f and h, and where they pinch together, g is pinched with them. The art is choosing good outer bounds — usually built from an inequality you already know, like $-1 \leq \cos x \leq 1$.

Example 6 (Applying the Squeeze Theorem)**.** Evaluate $\displaystyle\lim_{x \to 0} x \cos x$.

Solution. We cannot substitute usefully, because $\cos x$ oscillates. But we do know $\cos x$ is boxed in: $-1 \leq \cos x \leq 1$ for every x. Multiply this inequality through by $|x|$ (which is ≥ 0, so the inequalities keep their direction) to trap $x \cos x$:

$$-|x| \leq x \cos x \leq |x|.$$

The outer functions are $f(x) = -|x|$ and $h(x) = |x|$, and both are squeezed to 0:

$$\lim_{x \to 0} \big(-|x|\big) = 0 = \lim_{x \to 0} |x|.$$

Since $x \cos x$ is caught between two functions both heading to 0, the Squeeze Theorem forces

$$\lim_{x \to 0} x \cos x = 0.$$

Practice (work this out). Use the Squeeze Theorem to evaluate $\displaystyle\lim_{x \to 0} x^2 \sin \frac{1}{x}$. *(Hint: start from $-1 \leq \sin \frac{1}{x} \leq 1$.)*

Summary

- The **basic limits** $\lim_{x \to a} x = a$ and $\lim_{x \to a} c = c$, combined with the **limit laws** (sum, difference, constant multiple, product, quotient, power, root), let us compute limits exactly instead of guessing from tables.
- Each law requires the pieces to have their own limits; the quotient law also needs a nonzero denominator limit.
- For a **polynomial**, or a **rational function** where the denominator isn't zero, the limit is just the value: **direct substitution**. Try this first.
- When substitution gives the **indeterminate form** $0/0$, rewrite before substituting: **factor and cancel** for polynomials, multiply by the **conjugate** for square roots. Canceling is legal because the limit ignores the single point $x = a$.
- The **Squeeze Theorem**: if $f \leq g \leq h$ near a and f, h share the same limit L at a, then $g \to L$ too — the tool for trig limits we can't factor.
- Next time: continuity, and what it means for a function to have no breaks.