Continuity

Section 2.4

Calculus I — Lecture Notes

September 8, 2026

The Idea: No Breaks

A continuous function is one you can draw without lifting your pencil.

Last time we learned to compute limits. Today we use them to make precise an idea you already have an intuition for: a function that has *no breaks*. Many functions can be traced with a pencil from left to right without ever lifting it off the page — these are the **continuous** functions. Others have a hole, a jump, or shoot off to infinity somewhere; at those spots the graph breaks and we must lift the pencil. Our job is to turn "no break at this point" into three checkable conditions, then to classify the ways a break can happen, and finally to cash in continuity for a genuinely useful theorem.

Continuity at a Point

What could go wrong at a point a? Picture a graph with a break at a and ask what failed. Three separate things can go wrong, and ruling all three out is exactly what we need.

First, maybe $f(a)$ is simply **undefined** — there is a hole in the graph and no point sits above a at all. So we need $f(a)$ to exist. Second, even if $f(a)$ is defined, the graph might jump right at a, so the two-sided limit doesn't exist. So we need $\lim_{x\to a} f(x)$ to exist. Third, even if both of those hold, the point $f(a)$ might sit off on its own, away from where the limit is heading. So we need the limit to actually *equal* $f(a)$. Putting all three together:

Definition 1. A function f is **continuous at a point** a if and only if all three of the following hold:

(i) $f(a)$ is defined;

(ii) $\lim\limits_{x\to a} f(x)$ exists;

(iii) $\lim\limits_{x\to a} f(x) = f(a)$.

If f fails any one of these, we say f is **discontinuous at** a.

Notice condition (iii) quietly contains the other two — for the equation to make sense both sides must exist. Still, it is cleanest to check them in order: does the point exist, does the limit exist, do they match. The next three examples show each condition failing in turn.

Example 1 (Condition (i) fails)**.** Determine whether $f(x) = \dfrac{x^2-4}{x-2}$ is continuous at $x = 2$.

Solution. Start with condition (i): try to compute $f(2)$. We get

$$f(2) = \frac{2^2-4}{2-2} = \frac{0}{0},$$

which is undefined. Condition (i) already fails, so we can stop: f is **discontinuous at** $x = 2$. The graph is the line $y = x + 2$ with a hole punched at $(2, 4)$.

Example 2 (Condition (ii) fails)**.** Determine whether

$$f(x) = \begin{cases} -x^2 + 4, & x \le 3, \\ 4x - 8, & x > 3 \end{cases}$$

is continuous at $x = 3$.

Solution. Condition (i): the rule for $x \le 3$ applies at 3, so

$$f(3) = -(3)^2 + 4 = -5,$$

which is defined. Condition (ii): check the two one-sided limits.

$$\lim_{x \to 3^-} f(x) = -(3)^2 + 4 = -5, \qquad \lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$

The one-sided limits disagree, so $\lim_{x \to 3} f(x)$ does not exist. Condition (ii) fails, and f is **discontinuous at** $x = 3$.

Example 3 (All three conditions hold)**.** Determine whether

$$f(x) = \begin{cases} x^2 + 1, & x \ne 0, \\ 1, & x = 0 \end{cases}$$

is continuous at $x = 0$.

Solution. Condition (i): $f(0) = 1$, defined. Condition (ii): for x near (but not equal to) 0 we use $x^2 + 1$, so

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 + 1) = 0 + 1 = 1,$$

which exists. Condition (iii): compare, and

$$\lim_{x \to 0} f(x) = 1 = f(0).$$

All three conditions hold, so f is **continuous at** $x = 0$.

Two big families are continuous everywhere they are defined, and it saves enormous work to know this: **polynomials are continuous at every real number**, and a **rational function is continuous everywhere except where its denominator is zero**. (Trigonometric functions are continuous on their domains too.) So for these functions, "where is it continuous?" is answered just by finding the domain.

Practice (work this out). Using the three conditions, determine whether

$$f(x) = \begin{cases} 2x + 1, & x < 1, \\ 2, & x = 1, \\ -x + 4, & x > 1 \end{cases}$$

is continuous at $x = 1$. If not, state which condition fails.

Types of Discontinuities

When a function *does* break at a, the break has a shape. There are three common kinds, and we tell them apart entirely by the limit at a.

Definition 2. Suppose f is discontinuous at a. Then f has

(1) a **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists (i.e. equals a real number L) — the graph has a single hole you could "fill in";

(2) a **jump discontinuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but are unequal — the two sides don't meet up;

(3) an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ and/or $\lim_{x\to a^+} f(x) = \pm\infty$ — the graph runs off along a vertical asymptote.

The strategy is always the same: we already know the function breaks at a; now compute the limit at a and read off which case you are in.

Example 4 (A removable discontinuity)**.** Classify the discontinuity of $f(x) = \dfrac{x^2-4}{x-2}$ at $x = 2$.

Solution. We saw $f(2)$ is undefined, so f breaks at 2. Now take the limit, cancelling the common factor:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

The limit exists (it equals 4), so this is a **removable discontinuity**. Defining $f(2) = 4$ would plug the hole and make f continuous there.

Example 5 (A jump discontinuity)**.** Classify the discontinuity of

$$f(x) = \begin{cases} -x^2+4, & x \le 3, \\ 4x-8, & x > 3 \end{cases}$$

at $x = 3$.

Solution. From before, the one-sided limits are

$$\lim_{x\to 3^-} f(x) = -5, \qquad \lim_{x\to 3^+} f(x) = 4.$$

Both are real numbers, but they disagree. This is a **jump discontinuity** at $x = 3$: the graph steps from height -5 up to height 4.

Example 6 (An infinite discontinuity)**.** Determine whether $f(x) = \dfrac{x+2}{x+1}$ is continuous at $x = -1$, and if not, classify the discontinuity.

Solution. At $x = -1$ the denominator is 0, so $f(-1)$ is undefined and f breaks there. Examine the one-sided limits. As $x \to -1$ the numerator approaches $1 > 0$, while the denominator $x+1$ approaches 0 — negative from the left, positive from the right:

$$\lim_{x\to -1^-} \frac{x+2}{x+1} = -\infty, \qquad \lim_{x\to -1^+} \frac{x+2}{x+1} = +\infty.$$

Because the limits run off to infinity, this is an **infinite discontinuity** at $x = -1$; the line $x = -1$ is a vertical asymptote.

Practice (work this out). For each function, find where it is discontinuous and classify the discontinuity as removable, jump, or infinite:

$$\text{(a)}\quad f(x) = \frac{x^2-9}{x-3}, \qquad\qquad \text{(b)}\quad g(x) = \frac{5}{x-2}.$$

Continuity on an Interval

So far we have checked one point at a time. Usually we want to say a function is continuous over a whole **interval** — traceable, without lifting the pencil, between any two of its points. For an open interval this just means continuous at every point inside. Closed endpoints need a little care, because at a left endpoint there is nothing to the left to approach from, and at a right endpoint nothing to the right. So we only ask for the one-sided limit that lives inside the interval.

Definition 3. f is **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$, and **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$. A function is **continuous over the closed interval** $[a, b]$ if it is continuous at every point of (a, b), continuous from the right at a, and continuous from the left at b.

Example 7 (Continuity of a rational function on intervals)**.** State the intervals over which $f(x) = \dfrac{x-1}{x^2+2x}$ is continuous.

Solution. This is a rational function, so it is continuous everywhere except where the denominator vanishes. Factor: $x^2 + 2x = x(x+2)$, which is zero at $x = 0$ and $x = -2$. The domain is everything else, so f is continuous on each of

$$(-\infty, -2), \qquad (-2, 0), \qquad (0, \infty).$$

Example 8 (A closed interval from a root function)**.** State the interval over which $f(x) = \sqrt{4 - x^2}$ is continuous.

Solution. We need $4 - x^2 \geq 0$, i.e. $-2 \leq x \leq 2$, so the domain is $[-2, 2]$. On the open interval $(-2, 2)$ the limit laws give $\lim_{x \to a} \sqrt{4 - x^2} = \sqrt{4 - a^2} = f(a)$, so f is continuous there. At the endpoints we check the one interior side:

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 = f(-2), \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0 = f(2),$$

so f is continuous from the right at -2 and from the left at 2. Therefore f is continuous over the closed interval $[-2, 2]$.

Practice (work this out). State the interval(s) over which $f(x) = \sqrt{x+3}$ is continuous.

The Intermediate Value Theorem

Continuity is not just tidy bookkeeping — it buys us a theorem with real force. If a continuous function starts below some height and ends above it, then somewhere in between it must *hit* that height exactly. It cannot skip over a value, because skipping would require lifting the pencil.

Definition 4 (Intermediate Value Theorem)**.** Let f be continuous over a closed, bounded interval $[a, b]$. If z is any number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ with $f(c) = z$.

The most common use: to show an equation has a solution. If f is continuous and $f(a)$ and $f(b)$ have *opposite signs*, then $z = 0$ lies between them, so there must be a c with $f(c) = 0$ — a root trapped between a and b.

Example 9 (Trapping a root)**.** Show that $f(x) = x - \cos x$ has at least one zero.

Solution. As a difference of the continuous functions x and $\cos x$, f is continuous everywhere, hence continuous on any closed interval. We hunt for two inputs where f has opposite signs. Try $x = 0$ and $x = \frac{\pi}{2}$:

$$f(0) = 0 - \cos 0 = -1 < 0, \qquad f\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2} - \cos\tfrac{\pi}{2} = \tfrac{\pi}{2} - 0 = \tfrac{\pi}{2} > 0.$$

Since f is continuous on $\left[0, \frac{\pi}{2}\right]$ and 0 lies between $f(0) = -1$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, the Intermediate Value Theorem gives a c in $\left[0, \frac{\pi}{2}\right]$ with $f(c) = 0$. So $f(x) = x - \cos x$ has at least one zero.

Two cautions, because the theorem is often misread. First, it *requires* continuity: for $f(x) = \frac{1}{x}$ on $[-1, 1]$ we have $f(-1) = -1 < 0$ and $f(1) = 1 > 0$, yet there is no zero in between — the function isn't continuous over $[-1, 1]$ (it blows up at 0), so the theorem simply does not apply. Second, the theorem only *guarantees* a value is hit; it never says a value is missed. If $f(a)$ and $f(b)$ are both positive, the theorem tells you nothing about whether there is a zero — there might be one anyway, as with $f(x) = (x-1)^2$ on $[0, 2]$.

Practice (work this out). Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Summary

- f is **continuous at** a when all three hold: $f(a)$ is defined, $\lim_{x\to a} f(x)$ exists, and the two are equal. Fail any one and f is discontinuous at a.
- **Polynomials** are continuous everywhere; **rational** functions are continuous except where the denominator is zero; trig functions are continuous on their domains.
- **Three discontinuities**, told apart by the limit at a: *removable* (limit exists — a fillable hole), *jump* (one-sided limits exist but differ), *infinite* (limit is $\pm\infty$ — a vertical asymptote).
- **Continuity on an interval** means continuous at every interior point, using one-sided continuity at closed endpoints.
- **Intermediate Value Theorem:** a function continuous on $[a, b]$ hits every value between $f(a)$ and $f(b)$. Opposite signs at the endpoints trap a root — but continuity is required, and the theorem only guarantees values are hit, never that they are missed.